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Existence of nonoscillatory solutions of second-order nonlinear neutral differential equations with distributed deviating arguments

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ABSTRACT

Some sufficient conditions are provided for the existence of nonoscillatory solutions of nonlinear second-order neutral differential equations with distributed deviating arguments. The main tool for proving our results is the Banach contraction principle. Two examples are given to illustrate the effectiveness of our results.

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1. Introduction

The purpose of this article is to study the second-order neutral nonlinear differential equations with distributed deviating arguments of the form

$$\left(r(t)\left(x(t)-p(t)x(t-\tau)\right)'\right)'+\int_{a_1}^{b_1} f_1(t,x(\sigma_1(t,\xi)))\mathrm{d}\xi - \int_{a_2}^{b_2} f_2(t,x(\sigma_2(t,\xi)))\mathrm{d}\xi = g(t) \tag{1}$$

and

$$\left(r(t)\left(x(t)-\int_a^b P(t,\xi)x(t-\xi)\mathrm{d}\xi\right)'\right)' + \int_{a_1}^{b_1} f_1(t,x(\sigma_1(t,\xi)))\mathrm{d}\xi - \int_{a_2}^{b_2} f_2(t,x(\sigma_2(t,\xi)))\mathrm{d}\xi = g(t), \tag{2}$$

where $g\in C([t_0,\infty),\mathbb{R})$, $\tau>0$, $r\in C([t_0,\infty),(0,\infty))$, $p\in C([t_0,\infty),\mathbb{R})$, $P\in C([t_0,\infty)\times[a,b],\mathbb{R})$ for $0<a<b$, and $\sigma_i\in C([t_0,\infty)\times[a_i,b_i],\mathbb{R})$ with $\lim_{t\to\infty}\sigma_i(t,\xi)=\infty$ for $\xi\in[a_i,b_i]$, $a_i\geq 0$, $i=1,2$.

Throughout this study, we assume that $f_i\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ is nondecreasing in the second variable, $i=1,2$, $xf_i(t,x)>0$ for $x\neq 0$, $i=1,2$, and satisfies

$$|f_i(t,x)-f_i(t,y)|\leq q_i(t)|x-y| \quad \text{for } t\in[t_0,\infty) \tag{3}$$

and $x,y\in[e,f]$, where $q_i\in C([t_0,\infty),(0,\infty))$, $i=1,2$, and $[e,f]$ $(0<e<f$ or $e<f<0)$ is any closed interval. Furthermore, suppose that

$$\int_{t_0}^{\infty}\int_s^{\infty}\frac{q_i(u)}{r(s)}\mathrm{d}u\,\mathrm{d}s<\infty, \quad i=1,2, \tag{4}$$

$$\int_{t_0}^{\infty}\int_s^{\infty}\frac{|f_i(u,d)|}{r(s)}\mathrm{d}u\,\mathrm{d}s<\infty \quad \text{for some } \mathrm{d}\neq 0, \quad i=1,2, \tag{5}$$

$$\int_{t_0}^{\infty}\int_s^{\infty}\frac{|g(u)|}{r(s)}\mathrm{d}u\,\mathrm{d}s<\infty \tag{6}$$

hold.

In recent years, there have been many studies concerning the oscillatory and nonoscillatory behaviour of neutral differential equations (see [1–16] and references cited therein). For some studies of the qualitative analysis of Volterra integro-differential equations and a variable delay system of differential equations of second order, we refer the reader to [17, 18] and references cited therein. For example, in 2010, Candan and Dahiya [3] considered the existence of nonoscillatory solutions of first and second-order neutral equations of the form

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[x(t)+P(t)x(t-\tau)\right]+\int_a^b q_1(t,\xi)x(t-\xi)\mathrm{d}\xi - \int_c^d q_2(t,\mu)x(t-\mu)\mathrm{d}\mu=0,$$

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in 2016, Candan [4] investigated nonoscillatory solutions of higher order neutral differential equations of form

$$\left[r(t)\left[\left[x(t)-\int_a^b p_2(t,\xi)x(t-\xi)\mathrm{d}\xi\right]^{(n-1)}\right]^{\gamma}\right]' + (-1)^n\int_c^d Q_2(x,\xi)G(x(t-\xi))\mathrm{d}\xi = 0,$$

and in 2019, Çına et al. [8] studied the existence of nonoscillatory solutions of a nonlinear second-order neutral differential equation with forcing term

$$\left(r(t)\left(x(t)-p(t)x(t-\tau)\right)'\right)' + f_1(t,x(\sigma_1(t))) - f_2(t,x(\sigma_2(t))) = g(t).$$

Motivated by the above-mentioned studies, the aim of this paper is to give some sufficient conditions for the existence of nonoscillatory solutions of (1) and (2), more general than the latter equation, by using the Banach contraction principle.

Let $T_0 = \min\{t_1-\tau, \inf_{t\geq t_1}\min_{\xi\in[a_1,b_1]}\sigma_1(t,\xi), \inf_{t\geq t_1}\min_{\xi\in[a_2,b_2]}\sigma_2(t,\xi)\}$ for $t_1 \geq t_0$. By a solution of Equation (1), we mean a function $x \in C([T_0,\infty),\mathbb{R})$ in the sense that both $x(t)-p(t)x(t-\tau)$ and $r(t)(x(t)-p(t)x(t-\tau))'$ are continuously differentiable on $[t_1,\infty]$ and such that Equation (1) is satisfied for $t \geq t_1$.

Let $T_1 = \min\{t_1-b, \inf_{t\geq t_1}\min_{\xi\in[a_1,b_1]}\sigma_1(t,\xi), \inf_{t\geq t_1}\min_{\xi\in[a_2,b_2]}\sigma_2(t,\xi)\}$ for $t_1 \geq t_0$. By a solution of Equation (2), we mean a function $x \in C([T_1,\infty),\mathbb{R})$ in the sense that both $x(t)-\int_a^b p(t,\xi)x(t-\xi)\mathrm{d}\xi$ and $r(t)(x(t)-\int_a^b p(t,\xi)x(t-\xi)\mathrm{d}\xi)'$ are continuously differentiable on $[t_1,\infty]$ and such that Equation (2) is satisfied for $t \geq t_1$.

As is customary, a solution of (1) or (2) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory.

2. Main results

We suppose throughout this paper that X is the set of all continuous and bounded functions on $[t_0,\infty)$ with the norm $\|x\| = \sup_{t\geq t_0}|x(t)| < \infty$.

Theorem 2.1: *Assume that* (3)–(6) *hold and* $0 \leq p(t) \leq p < 1$. *Then* (1) *has a bounded nonoscillatory solution.*

Proof: Suppose (5) holds with $d > 0$. A similar argument holds for $d < 0$. Let $N_2 = d$. Set

$$A = \{x \in X : N_1 \leq x(t) \leq N_2, \quad t \geq t_0\},$$

where N_1 and N_2 are positive constants such that

$$N_1 < (1-p)N_2.$$

It is clear that A is a closed, bounded and convex subset of X. In view of (4)–(6), we can choose a $t_1 > t_0$ sufficiently large such that $t-\tau \geq t_0$, $\sigma_i(t,\xi) \geq t_0$, $\xi \in [a_i,b_i]$, $i = 1,2$ for $t \geq t_1$ and

$$p + \int_{t_1}^{\infty}\int_s^{\infty}\frac{1}{r(s)}[(b_1-a_1)q_1(u) + (b_2-a_2)q_2(u)]\mathrm{d}u\,\mathrm{d}s \leq \theta_1 < 1, \tag{7}$$

where θ_1 is a constant,

$$\int_{t_1}^{\infty}\int_s^{\infty}\frac{1}{r(s)}[(b_1-a_1)f_1(u,d) + |g(u)|]\mathrm{d}u\,\mathrm{d}s \leq \alpha - N_1, \tag{8}$$

and

$$\int_{t_1}^{\infty}\int_s^{\infty}\frac{1}{r(s)}[(b_2-a_2)f_2(u,d) + |g(u)|]\mathrm{d}u\,\mathrm{d}s \leq (1-p)N_2 - \alpha, \tag{9}$$

where $\alpha \in (N_1,(1-p)N_2)$. Define a mapping $S : A \longrightarrow X$ as follows:

$$(Sx)(t) = \begin{cases} \alpha + p(t)x(t-\tau) - \displaystyle\int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty} \\ \times\left[\displaystyle\int_{a_1}^{b_1} f_1(u,x(\sigma_1(u,\xi)))\mathrm{d}\xi \right. \\ \left. -\displaystyle\int_{a_2}^{b_2} f_2(u,x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u)\right]\mathrm{d}u\,\mathrm{d}s, \\ \quad t \geq t_1 \\ (Sx)(t_1), \quad t_0 \leq t \leq t_1. \end{cases}$$

It is clear that Sx is continuous. For every $x \in A$ and $t \geq t_1$, by (9), we get

$$\begin{aligned}(Sx)(t) &= \alpha + p(t)x(t-\tau) \\ &\quad - \int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty}\left[\int_{a_1}^{b_1} f_1(u,x(\sigma_1(u,\xi)))\mathrm{d}\xi \right. \\ &\quad \left. - \int_{a_2}^{b_2} f_2(u,x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u)\right]\mathrm{d}u\,\mathrm{d}s \\ &\leq \alpha + pN_2 + \int_{t_1}^{\infty}\int_s^{\infty}\frac{1}{r(s)}[(b_2-a_2)f_2(u,d) \\ &\quad + |g(u)|]\mathrm{d}u\,\mathrm{d}s \\ &\leq N_2\end{aligned}$$

and taking (8) into account, we have

$$\begin{aligned}(Sx)(t) &= \alpha + p(t)x(t-\tau) - \int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty} \\ &\quad \times\left[\int_{a_1}^{b_1} f_1(u,x(\sigma_1(u,\xi)))\mathrm{d}\xi \right. \\ &\quad \left. - \int_{a_2}^{b_2} f_2(u,x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u)\right]\mathrm{d}u\,\mathrm{d}s \\ &\geq \alpha - \int_{t_1}^{\infty}\int_s^{\infty}\frac{1}{r(s)}[(b_1-a_1)f_1(u,d) \\ &\quad + |g(u)|]\mathrm{d}u\,\mathrm{d}s \\ &\geq N_1.\end{aligned}$$

Thus $SA \subset A$. Now we show that S is a contraction mapping on A. In fact, for $x, y \in A$ and $t \geq t_1$, in view of (3) and (7), we have

$$\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\leq p|x(t-\tau) - y(t-\tau)| \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_1}^{b_1} |f_1(u, x(\sigma_1(u,\xi))) \\
&\quad - f_1(u, y(\sigma_1(u,\xi)))|\mathrm{d}\xi\, \mathrm{d}u\, \mathrm{d}s \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_2}^{b_2} |f_2(u, x(\sigma_2(u,\xi))) \\
&\quad - f_2(u, y(\sigma_2(u,\xi)))|\mathrm{d}\xi\, \mathrm{d}u\, \mathrm{d}s \\
&\leq p|x(t-\tau) - y(t-\tau)| \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_1}^{b_1} q_1(u)|x(\sigma_1(u,\xi)) \\
&\quad - y(\sigma_1(u,\xi))|\mathrm{d}\xi\, \mathrm{d}u\, \mathrm{d}s \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_2}^{b_2} q_2(u)|x(\sigma_2(u,\xi)) \\
&\quad - y(\sigma_2(u,\xi))|\mathrm{d}\xi\, \mathrm{d}u\, \mathrm{d}s \\
&\leq \|x - y\| \left[p + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} [(b_1 - a_1) q_1(u) \right. \\
&\quad \left. + (b_2 - a_2) q_2(u)]\mathrm{d}u\, \mathrm{d}s \right] \\
&\leq \theta_1 \|x - y\|.
\end{aligned}$$

This implies that $\|Sx - Sy\| \leq \theta_1 \|x - y\|$. Since $\theta_1 < 1$, S is a contraction mapping on A. Consequently, S has the unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof. ■

Theorem 2.2: *Assume that* (3)–(6) *hold and* $1 < p_1 \leq p(t) \leq p_2 < \infty$. *Then* (1) *has a bounded nonoscillatory solution.*

Proof: Suppose (5) holds with $d > 0$, the case $d < 0$ can be treated similarly. Let $N_4 = d$. Set

$$A = \{x \in X : N_3 \leq x(t) \leq N_4, \quad t \geq t_0\},$$

where N_3 and N_4 are positive constants such that

$$p_2 N_3 < (p_1 - 1) N_4.$$

It is obvious that A is a closed, bounded and convex subset of X. Because of (4)–(6), we can choose a $t_1 > t_0$ sufficiently large such that $\sigma_i(t + \tau, \xi) \geq t_0$, $\xi \in [a_i, b_i]$, $i = 1, 2$ for $t \geq t_1$ and

$$\frac{1}{p_1} \left[1 + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} [(b_1 - a_1) q_1(u) + (b_2 - a_2) q_2(u)]\mathrm{d}u\, \mathrm{d}s \right] \leq \theta_2 < 1, \tag{10}$$

where θ_2 is a constant,

$$\begin{aligned}
&\int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} [(b_1 - a_1) f_1(u, d) + |g(u)|]\mathrm{d}u\, \mathrm{d}s \\
&\quad \leq (p_1 - 1) N_4 - \alpha
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} [(b_2 - a_2) f_2(u, d) + |g(u)|]\mathrm{d}u\, \mathrm{d}s \\
&\quad \leq \alpha - p_2 N_3,
\end{aligned} \tag{12}$$

where $\alpha \in (p_2 N_3, (p_1 - 1) N_4)$. Consider the mapping $S : A \longrightarrow X$ defined by

$$(Sx)(t) = \begin{cases} \dfrac{1}{p(t+\tau)} \left[\alpha + x(t+\tau) + \displaystyle\int_{t+\tau}^\infty \frac{1}{r(s)} \int_s^\infty \right. \\ \times \left[\displaystyle\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi)))\mathrm{d}\xi \right. \\ \left. \left. - \displaystyle\int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u) \right] \mathrm{d}u\, \mathrm{d}s \right], \\ \quad t \geq t_1 \\ (Sx)(t_1), \ t_0 \leq t \leq t_1. \end{cases}$$

It is obvious that Sx is continuous. For every $x \in A$ and $t \geq t_1$, by (11), we have

$$\begin{aligned}
(Sx)(t) &= \frac{1}{p(t+\tau)} \left[\alpha + x(t+\tau) + \int_{t+\tau}^\infty \frac{1}{r(s)} \int_s^\infty \right. \\
&\quad \times \left[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi)))\mathrm{d}\xi \right. \\
&\quad \left. \left. - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u) \right] \mathrm{d}u\, \mathrm{d}s \right] \\
&\leq \frac{1}{p_1} \left[\alpha + N_4 + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} [(b_1 - a_1) f_1(u, d) \right. \\
&\quad \left. + |g(u)|]\mathrm{d}u\, \mathrm{d}s \right] \\
&\leq N_4
\end{aligned}$$

and from (12), we obtain

$$\begin{aligned}
(Sx)(t) &= \frac{1}{p(t+\tau)} \left[\alpha + x(t+\tau) + \int_{t+\tau}^\infty \frac{1}{r(s)} \int_s^\infty \right. \\
&\quad \times \left[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi)))\mathrm{d}\xi \right. \\
&\quad \left. \left. - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u) \right] \mathrm{d}u\, \mathrm{d}s \right] \\
&\geq \frac{1}{p_2} \left[\alpha - \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} [(b_2 - a_2) f_2(u, d) \right. \\
&\quad \left. + |g(u)|]\mathrm{d}u\, \mathrm{d}s \right] \\
&\geq N_3.
\end{aligned}$$

Thus $SA \subset A$. Finally, for $x, y \in A$ and $t \geq t_1$, in view of (3) and (10), we have

$$\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\leq \frac{1}{p(t+\tau)}\Bigg[|x(t+\tau) - y(t+\tau)| \\
&+ \int_{t+\tau}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \int_{a_1}^{b_1} |f_1(u, x(\sigma_1(u,\xi))) \\
&- f_1(u, y(\sigma_1(u,\xi)))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \\
&+ \int_{t+\tau}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \int_{a_2}^{b_2} |f_2(u, x(\sigma_2(u,\xi))) \\
&- f_2(u, y(\sigma_2(u,\xi)))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\leq \frac{1}{p_1}\Bigg[|x(t+\tau) - y(t+\tau)| \\
&+ \int_{t+\tau}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \int_{a_1}^{b_1} q_1(u) |x(\sigma_1(u,\xi)) \\
&- y(\sigma_1(u,\xi))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \\
&+ \int_{t+\tau}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \int_{a_2}^{b_2} q_2(u) |x(\sigma_2(u,\xi)) \\
&- y(\sigma_2(u,\xi))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\leq \frac{\|x - y\|}{p_1}\Bigg[1 + \int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)} \\
&\times [(b_1 - a_1) q_1(u) + (b_2 - a_2) q_2(u)] \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\leq \theta_2 \|x - y\|.
\end{aligned}$$

This implies that $\|Sx - Sy\| \leq \theta_2 \|x - y\|$ with $\theta_2 < 1$. Hence, S is a contraction mapping on A. Consequently, S has the unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). Thus, the theorem is proved. ■

Theorem 2.3: *Assume that* (3)–(6) *hold and* $-1 < -p \leq p(t) \leq 0$. *Then* (1) *has a bounded nonoscillatory solution.*

Proof: Suppose (5) holds with $d > 0$. A similar argument holds for $d < 0$. Let $N_6 = d$. Set

$$A = \{x \in X : N_5 \leq x(t) \leq N_6, \quad t \geq t_0\},$$

where N_5 and N_6 are positive constants such that

$$N_5 + pN_6 < N_6.$$

It is obvious that A is a closed, bounded and convex subset of X. In view of (4)–(6), we can choose a $t_1 > t_0$ sufficiently large such that $t - \tau \geq t_0$, $\sigma_i(t, \xi) \geq t_0$, $\xi \in [a_i, b_i]$, $i = 1, 2$ for $t \geq t_1$ and

$$\begin{aligned}
&p + \int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)} [(b_1 - a_1) q_1(u) + (b_2 - a_2) q_2(u)] \\
&\quad \times \mathrm{d}u \, \mathrm{d}s \leq \theta_3 < 1, \qquad (13)
\end{aligned}$$

where θ_3 is a constant,

$$\begin{aligned}
&\int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)} [(b_1 - a_1) f_1(u, d) + |g(u)|] \mathrm{d}u \, \mathrm{d}s \\
&\quad \leq \alpha - N_5 - pN_6, \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)} [(b_2 - a_2) f_2(u, d) + |g(u)|] \mathrm{d}u \, \mathrm{d}s \\
&\quad \leq N_6 - \alpha, \qquad (15)
\end{aligned}$$

where $\alpha \in (N_5 + pN_6, N_6)$. Define a mapping $S : A \longrightarrow X$ as follows:

$$(Sx)(t) = \begin{cases}
\alpha + p(t) x(t - \tau) - \int_{t}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} [\int_{a_1}^{b_1} \\
\times f_1(u, x(\sigma_1(u,\xi))) \mathrm{d}\xi \\
- \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi))) \mathrm{d}\xi - g(u)] \mathrm{d}u \, \mathrm{d}s, \\
t \geq t_1 \\
(Sx)(t_1), \quad t_0 \leq t \leq t_1.
\end{cases}$$

It is clear that Sx is continuous. For every $x \in A$ and $t \geq t_1$, from (15), we have

$$\begin{aligned}
(Sx)(t) &= \alpha + p(t) x(t - \tau) - \int_{t}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \\
&\quad \times \Bigg[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi))) \mathrm{d}\xi \\
&\quad - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi))) \mathrm{d}\xi - g(u)\Bigg] \mathrm{d}u \, \mathrm{d}s \\
&\leq \alpha + \int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)} [(b_2 - a_2) f_2(u, d) \\
&\quad + |g(u)|] \mathrm{d}u \, \mathrm{d}s \\
&\leq N_6
\end{aligned}$$

and by using (14), we have

$$\begin{aligned}
(Sx)(t) &= \alpha + p(t) x(t - \tau) - \int_{t}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \\
&\quad \times \Bigg[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi))) \mathrm{d}\xi \\
&\quad - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi))) \mathrm{d}\xi - g(u)\Bigg] \mathrm{d}u \, \mathrm{d}s \\
&\geq \alpha - pN_6 - \int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)} [(b_1 - a_1) f_1(u, d) \\
&\quad + |g(u)|] \mathrm{d}u \, \mathrm{d}s \\
&\geq N_5.
\end{aligned}$$

Thus $SA \subset A$. We shall show that S is a contraction mapping on A. In fact, for $x, y \in A$ and $t \geq t_1$, in view of (3)

and (13), we have

$$\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\le p|x(t-\tau) - y(t-\tau)| \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_1}^{b_1} |f_1(u, x(\sigma_1(u,\xi))) \\
&\quad - f_1(u, y(\sigma_1(u,\xi)))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_2}^{b_2} |f_2(u, x(\sigma_2(u,\xi))) \\
&\quad - f_2(u, y(\sigma_2(u,\xi)))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \\
&\le p|x(t-\tau) - y(t-\tau)| \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_1}^{b_1} q_1(u)|x(\sigma_1(u,\xi)) \\
&\quad - y(\sigma_1(u,\xi))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \\
&\quad + \int_t^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_2}^{b_2} q_2(u)|x(\sigma_2(u,\xi)) \\
&\quad - y(\sigma_2(u,\xi))| \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}s \\
&\le \|x - y\| \Bigg[p + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} \\
&\quad \times [(b_1 - a_1)q_1(u) + (b_2 - a_2)q_2(u)] \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\le \theta_3 \|x - y\|.
\end{aligned}$$

This implies that $\|Sx - Sy\| \le \theta_3 \|x - y\|$. Since $\theta_3 < 1$, S is a contraction mapping on A. Consequently, S has the unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof. ■

Theorem 2.4: *Assume that* (3)–(6) *hold and* $-\infty < -p_1 \le p(t) \le -p_2 < -1$. *Then* (1) *has a bounded nonoscillatory solution.*

Proof: Suppose (5) holds with $d > 0$, the case $d < 0$ can be treated similarly. Let $N_8 = d$. Set

$$A = \{x \in X : N_7 \le x(t) \le N_8, \quad t \ge t_0\},$$

where N_7 and N_8 are positive constants such that

$$p_1 N_7 + N_8 < p_2 N_8.$$

It is obvious that A is a closed, bounded and convex subset of X. In view of (4)–(6), there exists a $t_1 > t_0$ sufficiently large such that $\sigma_i(t + \tau, \xi) \ge t_0$, $\xi \in [a_i, b_i]$, $i = 1, 2$ for $t \ge t_1$ and

$$\frac{1}{p_2} \Bigg[1 + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} \big[(b_1 - a_1)q_1(u) + (b_2 - a_2)q_2(u) \big] \mathrm{d}u \, \mathrm{d}s \Bigg] \le \theta_4 < 1, \tag{16}$$

where θ_4 is a constant,

$$\begin{aligned}
&\int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} \big[(b_1 - a_1)f_1(u,d) + |g(u)| \big] \mathrm{d}u \, \mathrm{d}s \\
&\quad \le \alpha - p_1 N_7 - N_8
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} \big[(b_2 - a_2)f_2(u,d) + |g(u)| \big] \mathrm{d}u \, \mathrm{d}s \\
&\quad \le p_2 N_8 - \alpha,
\end{aligned} \tag{18}$$

where $\alpha \in (p_1 N_7 + N_8, p_2 N_8)$. Consider the mapping $S : A \longrightarrow X$ defined by

$$(Sx)(t) = \begin{cases} -\dfrac{1}{p(t+\tau)} \Bigg[\alpha - x(t+\tau) - \displaystyle\int_{t+\tau}^\infty \frac{1}{r(s)} \int_s^\infty \\ \times \Bigg[\displaystyle\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi))) \mathrm{d}\xi \\ - \displaystyle\int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi))) \mathrm{d}\xi - g(u) \Bigg] \mathrm{d}u \, \mathrm{d}s \Bigg], \\ t \ge t_1 \\ (Sx)(t_1), \ t_0 \le t \le t_1. \end{cases}$$

It is obvious that Sx is continuous. For every $x \in A$ and $t \ge t_1$, from (18), we have

$$\begin{aligned}
(Sx)(t) &= -\frac{1}{p(t+\tau)} \Bigg[\alpha - x(t+\tau) \\
&\quad - \int_{t+\tau}^\infty \frac{1}{r(s)} \int_s^\infty \Bigg[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi))) \mathrm{d}\xi \\
&\quad - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi))) \mathrm{d}\xi - g(u) \Bigg] \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\le \frac{1}{p_2} \Bigg[\alpha + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} \\
&\quad \times \big[(b_2 - a_2)f_2(u,d) + |g(u)| \big] \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\le N_8
\end{aligned}$$

and from (17), we obtain

$$\begin{aligned}
(Sx)(t) &= -\frac{1}{p(t+\tau)} \Bigg[\alpha - x(t+\tau) - \int_{t+\tau}^\infty \frac{1}{r(s)} \int_s^\infty \\
&\quad \times \Bigg[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi))) \mathrm{d}\xi \\
&\quad - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi))) \mathrm{d}\xi - g(u) \Bigg] \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\ge \frac{1}{p_1} \Bigg[\alpha - N_8 - \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} \\
&\quad \times \big[(b_1 - a_1)f_1(u,d) + |g(u)| \big] \mathrm{d}u \, \mathrm{d}s \Bigg] \\
&\ge N_7.
\end{aligned}$$

Thus $SA \subset A$. In fact, for $x, y \in A$ and $t \ge t_1$, by using (3) and (16), we have

$$\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\le \frac{1}{|p(t+\tau)|} \Bigg[|x(t+\tau) - y(t+\tau)| \\
&\quad + \int_{t+\tau}^\infty \frac{1}{r(s)} \int_s^\infty \int_{a_1}^{b_1} |f_1(u, x(\sigma_1(u,\xi)))
\end{aligned}$$

$$
\begin{aligned}
&- f_1(u, y(\sigma_1(u,\xi)))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s \\
&+ \int_{t+\tau}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \int_{a_2}^{b_2} |f_2(u, x(\sigma_2(u,\xi))) \\
&- f_2(u, y(\sigma_2(u,\xi)))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s\Big] \\
&\le \frac{1}{p_2}\Big[|x(t+\tau) - y(t+\tau)| \\
&+ \int_{t+\tau}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \int_{a_1}^{b_1} q_1(u)|x(\sigma_1(u,\xi)) \\
&- y(\sigma_1(u,\xi))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s \\
&+ \int_{t+\tau}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \int_{a_2}^{b_2} q_2(u)|x(\sigma_2(u,\xi)) \\
&- y(\sigma_2(u,\xi))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s\Big] \\
&\le \frac{\|x-y\|}{p_2}\Big[1 + \int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)} \\
&\times \Big[(b_1-a_1)q_1(u) + (b_2-a_2)q_2(u)]\mathrm{d}u\,\mathrm{d}s\Big] \\
&\le \theta_4\|x-y\|.
\end{aligned}
$$

This shows that $\|Sx - Sy\| \le \theta_4\|x-y\|$ with $\theta_4 < 1$. This implies that S is a contraction mapping on A. Thus, S has the unique fixed point $x \in A$ such that $Sx = x$, which is clearly a positive solution of (1). Hence the proof is complete. ■

Theorem 2.5: *Assume that* (3)–(6) *hold,* $P(t,\xi) \ge 0$ *and* $\int_a^b P(t,\xi)\mathrm{d}\xi \le p < 1$. *Then* (2) *has a bounded nonoscillatory solution.*

Proof: Suppose (5) holds with $d > 0$. A similar argument holds for $d < 0$. Let $N_{10} = d$. Set

$$A = \{x \in X : N_9 \le x(t) \le N_{10}, \quad t \ge t_0\},$$

where N_9 and N_{10} are positive constants such that

$$N_9 < (1-p)N_{10}.$$

It is obvious that A is a closed, bounded and convex subset of X. Because of (4)–(6), we can take a $t_1 > t_0$ sufficiently large such that $t - b \ge t_0$, $\sigma_i(t,\xi) \ge t_0$, $\xi \in [a_i, b_i]$, $i = 1, 2$ for $t \ge t_1$ and

$$
\begin{aligned}
p + \int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)}\big[&(b_1-a_1)q_1(u) \\
&+ (b_2-a_2)q_2(u)\big]\mathrm{d}u\,\mathrm{d}s \le \theta_5 < 1,
\end{aligned} \tag{19}
$$

where θ_5 is a constant,

$$
\begin{aligned}
&\int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)}\big[(b_1-a_1)f_1(u,d) + |g(u)|\big]\mathrm{d}u\,\mathrm{d}s \\
&\quad \le \alpha - N_9,
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
&\int_{t_1}^{\infty} \int_{s}^{\infty} \frac{1}{r(s)}\big[(b_2-a_2)f_2(u,d) + |g(u)|\big]\mathrm{d}u\,\mathrm{d}s \\
&\quad \le (1-p)N_{10} - \alpha,
\end{aligned} \tag{21}
$$

where $\alpha \in (N_9, (1-p)N_{10})$. Define a mapping $S : A \longrightarrow X$ as follows:

$$
(Sx)(t) = \begin{cases}
\alpha + \displaystyle\int_a^b P(t,\xi)x(t-\xi)\mathrm{d}\xi \\
\displaystyle - \int_t^{\infty} \frac{1}{r(s)} \int_s^{\infty} [\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi)))\mathrm{d}\xi \\
\displaystyle - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u)]\mathrm{d}u\,\mathrm{d}s, \\
t \ge t_1 \\
(Sx)(t_1),\ t_0 \le t \le t_1.
\end{cases}
$$

It is clear that Sx is continuous. For every $x \in A$ and $t \ge t_1$, from (21), we have

$$
\begin{aligned}
(Sx)(t) &= \alpha + \int_a^b P(t,\xi)x(t-\xi)\mathrm{d}\xi \\
&\quad - \int_t^{\infty} \frac{1}{r(s)} \int_s^{\infty} \Big[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi)))\mathrm{d}\xi \\
&\quad - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u)\Big]\mathrm{d}u\,\mathrm{d}s \\
&\le \alpha + pN_{10} + \int_{t_1}^{\infty} \int_s^{\infty} \frac{1}{r(s)} \\
&\quad \times \big[(b_2-a_2)f_2(u,d) + |g(u)|\big]\mathrm{d}u\,\mathrm{d}s \\
&\le N_{10}
\end{aligned}
$$

and by using (20), we get

$$
\begin{aligned}
(Sx)(t) &= \alpha + \int_a^b P(t,\xi)x(t-\xi)\mathrm{d}\xi \\
&\quad - \int_t^{\infty} \frac{1}{r(s)} \int_s^{\infty} \Big[\int_{a_1}^{b_1} f_1(u, x(\sigma_1(u,\xi)))\mathrm{d}\xi \\
&\quad - \int_{a_2}^{b_2} f_2(u, x(\sigma_2(u,\xi)))\mathrm{d}\xi - g(u)\Big]\mathrm{d}u\,\mathrm{d}s \\
&\ge \alpha - \int_{t_1}^{\infty} \int_s^{\infty} \frac{1}{r(s)} \\
&\quad \times \big[(b_1-a_1)f_1(u,d) + |g(u)|\big]\mathrm{d}u\,\mathrm{d}s \\
&\ge N_9.
\end{aligned}
$$

Thus $SA \subset A$. Now we show that S is a contraction mapping on A. In fact, for $x, y \in A$ and $t \ge t_1$, from (3) and (19), we have

$$
\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\le \int_a^b P(t,\xi)|x(t-\xi) - y(t-\xi)|\mathrm{d}\xi \\
&+ \int_t^{\infty} \frac{1}{r(s)} \int_s^{\infty} \int_{a_1}^{b_1} |f_1(u, x(\sigma_1(u,\xi))) \\
&- f_1(u, y(\sigma_1(u,\xi)))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^\infty \frac{1}{r(s)}\int_s^\infty\int_{a_2}^{b_2}|f_2(u,x(\sigma_2(u,\xi)))\\
&-f_2(u,y(\sigma_2(u,\xi)))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s\\
&\le\int_a^b P(t,\xi)|x(t-\xi)-y(t-\xi)|\mathrm{d}\xi\\
&+\int_t^\infty \frac{1}{r(s)}\int_s^\infty\int_{a_1}^{b_1} q_1(u)|x(\sigma_1(u,\xi))\\
&-y(\sigma_1(u,\xi))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s\\
&+\int_t^\infty \frac{1}{r(s)}\int_s^\infty\int_{a_2}^{b_2} q_2(u)|x(\sigma_2(u,\xi))\\
&-y(\sigma_2(u,\xi))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s\\
&\le\|x-y\|\Bigg[p+\int_{t_1}^\infty\int_s^\infty \frac{1}{r(s)}\\
&\times\big[(b_1-a_1)q_1(u)+(b_2-a_2)q_2(u)\big]\mathrm{d}u\,\mathrm{d}s\Bigg]\\
&\le\theta_5\|x-y\|.
\end{aligned}
$$

This implies that $\|Sx-Sy\|\le\theta_5\|x-y\|$. Since $\theta_5<1$, S is a contraction mapping on A. Consequently, S has the unique fixed point $x\in A$ such that $Sx=x$, which is obviously a positive solution of (2). This completes the proof. ■

Theorem 2.6: *Assume that* (3)–(6) *hold,* $P(t,\xi)\le 0$ *and* $-1<-p\le\int_a^b P(t,\xi)\mathrm{d}\xi$. *Then* (2) *has a bounded nonoscillatory solution.*

Proof: Suppose (5) holds with $d>0$, the case $d<0$ can be treated similarly. Let $N_{12}=d$. Set

$$A=\{x\in X: N_{11}\le x(t)\le N_{12},\quad t\ge t_0\},$$

where N_{11} and N_{12} are positive constants such that

$$pN_{12}+N_{11}<N_{12}.$$

It is clear that A is a closed, bounded and convex subset of X. By (4)–(6), we can take a $t_1>t_0$ sufficiently large such that $t-b\ge t_0$, $\sigma_i(t,\xi)\ge t_0$, $\xi\in[a_i,b_i]$, $i=1,2$ for $t\ge t_1$ and

$$
\begin{aligned}
p+\int_{t_1}^\infty\int_s^\infty\frac{1}{r(s)}\big[&(b_1-a_1)q_1(u)\\
&+(b_2-a_2)q_2(u)\big]\mathrm{d}u\,\mathrm{d}s\le\theta_6<1,
\end{aligned}
\tag{22}
$$

where θ_6 is a constant,

$$
\begin{aligned}
&\int_{t_1}^\infty\int_s^\infty\frac{1}{r(s)}\big[(b_1-a_1)f_1(u,d)+|g(u)|\big]\mathrm{d}u\,\mathrm{d}s\\
&\quad\le\alpha-pN_{12}-N_{11},
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&\int_{t_1}^\infty\int_s^\infty\frac{1}{r(s)}\big[(b_2-a_2)f_2(u,d)+|g(u)|\big]\mathrm{d}u\,\mathrm{d}s\\
&\quad\le N_{12}-\alpha,
\end{aligned}
\tag{24}
$$

where $\alpha\in(pN_{12}+N_{11},N_{12})$. Consider the mapping $S: A\longrightarrow X$ defined by

$$
(Sx)(t)=\begin{cases}
\alpha+\displaystyle\int_a^b P(t,\xi)x(t-\xi)\mathrm{d}\xi-\int_t^\infty\frac{1}{r(s)}\int_s^\infty\\
\times\Bigg[\displaystyle\int_{a_1}^{b_1}f_1(u,x(\sigma_1(u,\xi)))\mathrm{d}\xi\\
-\displaystyle\int_{a_2}^{b_2}f_2(u,x(\sigma_2(u,\xi)))\mathrm{d}\xi-g(u)\Bigg]\mathrm{d}u\,\mathrm{d}s,\\
t\ge t_1\\
(Sx)(t_1),\quad t_0\le t\le t_1.
\end{cases}
$$

It is obvious that Sx is continuous. For every $x\in A$ and $t\ge t_1$, by (24), we have

$$
\begin{aligned}
(Sx)(t)&=\alpha+\int_a^b P(t,\xi)x(t-\xi)\mathrm{d}\xi\\
&\quad-\int_t^\infty\frac{1}{r(s)}\int_s^\infty\Bigg[\int_{a_1}^{b_1}f_1(u,x(\sigma_1(u,\xi)))\mathrm{d}\xi\\
&\quad-\int_{a_2}^{b_2}f_2(u,x(\sigma_2(u,\xi)))\mathrm{d}\xi-g(u)\Bigg]\mathrm{d}u\,\mathrm{d}s\\
&\le\alpha+\int_{t_1}^\infty\int_s^\infty\frac{1}{r(s)}\\
&\quad\times\big[(b_2-a_2)f_2(u,d)+|g(u)|\big]\mathrm{d}u\,\mathrm{d}s\\
&\le N_{12}
\end{aligned}
$$

and taking (23) into account, we get

$$
\begin{aligned}
(Sx)(t)&=\alpha+\int_a^b P(t,\xi)x(t-\xi)\mathrm{d}\xi-\int_t^\infty\frac{1}{r(s)}\int_s^\infty\\
&\quad\times\Bigg[\int_{a_1}^{b_1}f_1(u,x(\sigma_1(u,\xi)))\mathrm{d}\xi\\
&\quad-\int_{a_2}^{b_2}f_2(u,x(\sigma_2(u,\xi)))\mathrm{d}\xi-g(u)\Bigg]\mathrm{d}u\,\mathrm{d}s\\
&\ge\alpha-pN_{12}-\int_{t_1}^\infty\int_s^\infty\frac{1}{r(s)}\\
&\quad\times\big[(b_1-a_1)f_1(u,d)+|g(u)|\big]\mathrm{d}u\,\mathrm{d}s\\
&\ge N_{11}.
\end{aligned}
$$

Thus $SA\subset A$. Now we show that S is a contraction mapping on A. In fact, for $x,y\in A$ and $t\ge t_1$, in view of (3) and (22), we have

$$
\begin{aligned}
|(Sx)(t)-(Sy)(t)|&\le\int_a^b(-P(t,\xi))|x(t-\xi)\\
&\quad-y(t-\xi)|\mathrm{d}\xi+\int_t^\infty\frac{1}{r(s)}\int_s^\infty\int_{a_1}^{b_1}\\
&\quad\times|f_1(u,x(\sigma_1(u,\xi)))-f_1(u,y(\sigma_1(u,\xi)))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s\\
&\quad+\int_t^\infty\frac{1}{r(s)}\int_s^\infty\int_{a_2}^{b_2}|f_2(u,x(\sigma_2(u,\xi)))\\
&\quad-f_2(u,y(\sigma_2(u,\xi)))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s
\end{aligned}
$$

$$\leq \int_a^b (-P(t,\xi))|x(t-\xi)-y(t-\xi)|\mathrm{d}\xi + \int_t^\infty \frac{1}{r(s)}\int_s^\infty \int_{a_1}^{b_1} q_1(u)|x(\sigma_1(u,\xi)) - y(\sigma_1(u,\xi))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s + \int_t^\infty \frac{1}{r(s)}\int_s^\infty \int_{a_2}^{b_2} q_2(u) \times |x(\sigma_2(u,\xi)) - y(\sigma_2(u,\xi))|\mathrm{d}\xi\,\mathrm{d}u\,\mathrm{d}s$$
$$\leq \|x-y\|\left[p + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)}[(b_1-a_1)q_1(u) + (b_2-a_2)q_2(u)]\mathrm{d}u\,\mathrm{d}s\right]$$
$$\leq \theta_6\|x-y\|.$$

This implies that $\|Sx - Sy\| \leq \theta_6\|x-y\|$ with $\theta_6 < 1$, and S is a contraction mapping on A. Consequently, S has the unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (2). Thus the theorem is proved. ■

Example 2.7: For $t > 4$, consider the equation

$$(\mathrm{e}^t(x(t) - \mathrm{e}^{-t}x(t-2))')' + \int_0^2 2\mathrm{e}^{-t+2}x(t-2\xi)\mathrm{d}\xi - \int_2^4 \mathrm{e}^{-t+2}x(t-\xi)\mathrm{d}\xi = 2\mathrm{e}^{2-t} - \mathrm{e}^{2-2t} + \mathrm{e}^{4-2t}. \tag{25}$$

Note that $r(t) = \mathrm{e}^t$, $p(t) = \mathrm{e}^{-t}$, $\tau = 2$, $\sigma_1(t,\xi) = t - 2\xi$, $\sigma_2(t,\xi) = t - \xi$, $f_1(t,x) = 2\mathrm{e}^{-t+2}x$, $f_2(t,x) = \mathrm{e}^{-t+2}x$ and $g(t) = 2\mathrm{e}^{2-t} - \mathrm{e}^{2-2t} + \mathrm{e}^{4-2t}$. We can check that all the conditions of Theorem 2.1 are satisfied. We note that $x(t) = \mathrm{e}^{-t} + 2$ is a nonoscillatory solution of (25).

Example 2.8: For $t > 4$, consider the equation

$$(\mathrm{e}^{t/2}(x(t) - \int_3^4 \mathrm{e}^{-t-2\xi}x(t-\xi)\mathrm{d}\xi)')' + \int_1^2 3\mathrm{e}^{-t}x(t-2\xi)\mathrm{d}\xi - \int_2^4 2\mathrm{e}^{-t}x(t-\xi)\mathrm{d}\xi = \frac{1}{4}\mathrm{e}^{-t/2-8} - \frac{1}{4}\mathrm{e}^{-t/2-6} + 3\mathrm{e}^{-3t/2} - \frac{15}{2}\mathrm{e}^{-5t/2} - \mathrm{e}^{-t} + \frac{1}{4}\mathrm{e}^{4-3t} - \frac{1}{4}\mathrm{e}^{8-3t}. \tag{26}$$

Equation (26) is a special case of (2) with $r(t) = \mathrm{e}^{t/2}$, $P(t,\xi) = \mathrm{e}^{-t-2\xi}$, $\sigma_1(t,\xi) = t - 2\xi$, $\sigma_2(t,\xi) = t - \xi$, $f_1(t,x) = 3\mathrm{e}^{-t}x$, $f_2(t,x) = 2\mathrm{e}^{-t}x$ and $g(t) = \frac{1}{4}\mathrm{e}^{-t/2-8} - \frac{1}{4}\mathrm{e}^{-t/2-6} + 3\mathrm{e}^{-3t/2} - \frac{15}{2}\mathrm{e}^{-5t/2} - \mathrm{e}^{-t} + \frac{1}{4}\mathrm{e}^{4-3t} - \frac{1}{4}\mathrm{e}^{8-3t}$. The conditions of Theorem 2.5 are clearly satisfied. It is obvious that $x(t) = \mathrm{e}^{-2t} + 1$ is a nonoscillatory solution of (26).

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