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Exploring Nonoscillatory Solutions in Second-Order Neutral Differential Equations with Distributed Deviating Arguments

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Highlights

- This paper focuses on nonoscillatory solutions of second-order neutral differential equations.
- Banach's Contraction Mapping principe was utilised in the theorem proofs.
- The existence of positive solutions was proved by showing the existence of a fixed point.

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Abstract

The occurrence of oscillation and non-oscillation is common across various models in real-world applications. For instance, impulsive partial neutral differential equations in mathematical biology and biomathematics often display both oscillatory and non-oscillatory solutions. In our investigation, we establish specific criteria that guarantee the presence of non-oscillatory solutions for variable coefficient nonlinear second-order neutral differential equations with distributed deviating arguments and a forcing term. In this work, we obtained an extension of some existing results in the literature. Using the Banach contraction principle, we obtained new sufficient conditions of existence conditions. The proof of positive solutions was provided by showing the existence of a fixed point. At the end of the paper, we give an example showing how we apply one of the theorems we have learnt.

1. INTRODUCTION

This study focuses on a second-order nonlinear neutral differential equation with distributed delays as the following form

$$(x(t)-p(t)x(t-\tau))''+\int_{a_1}^{b_1} f_1\left(t,x\left(\sigma_1(t,\xi)\right)\right)d\xi-\int_{a_2}^{b_2} f_2\left(t,x\left(\sigma_2(t,\xi)\right)\right)d\xi=g(t), \tag{1}$$

where $p,g\in C([t_0,\infty),\mathbb{R})$, $\tau>0$, $\sigma_i(t,\xi)\in C([t_0,\infty)\times[a_i,b_i],\mathbb{R})$ and $0\le a_i<b_i$, $i=1,2$.

In our article, we assume that $\sigma_i(t,\xi)\to\infty$ as $t\to\infty$ for $\xi\in[a_i,b_i]$, $i=1,2$ and $f_i(t,x)\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ is a nondecreasing in x for $i=1,2$ and satisfying $xf_i(t,x)>0$ for $x\neq 0$, $i=1,2$ and there exist a function $q_i\in C([t_0,\infty),\mathbb{R}^+)$, $i=1,2$ such that

$$|f_i(t,x)-f_i(t,y)|\le q_i(t)|x-y| \text{ for } t\in[t_0,\infty) \text{ and } x,y\in[a,b], \tag{2}$$

where $[a,b]$ $(0<a<b$ or $a<b<0)$ is any closed interval.

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Let us also assume that

$$\int_{t_0}^{\infty} sq_i(s)ds < \infty, \qquad i = 1,2, \tag{3}$$

$$\int_{t_0}^{\infty} s|f_i(s,d)|ds < \infty, \quad d \neq 0, \ i = 1,2 \tag{4}$$

$$\int_{t_0}^{\infty} s|g(s)|ds < \infty. \tag{5}$$

Various authors have examined the nonoscillatory behavior of solutions of neutral differential equations in the literature. This article is inspired by the research conducted by Yang, Zhang, and Ge in [1], focusing on the presence of nonoscillatory solutions for second-order differential equations represented by the following form

$$(x(t) - p(t)x(t-\tau))'' + f_1(t, x(\sigma_1(t))) - f_2(t, x(\sigma_2(t))) = 0\,. \tag{6}$$

In [2], Candan and Dahiya discuss the presence of non-oscillatory solutions for first and second order neutral differential equations of the type

$$\frac{d^k}{dt^k}[x(t) + P(t)x(t-\tau)] + \int_a^b q_1(t,\xi)x(t-\xi)d\xi - \int_c^d q_2(t,\mu)x(t-\mu)d\mu = 0. \tag{7}$$

Also in [3] Şenel and Çına considered the second-order neutral nonlinear differential equation with distributed deviating arguments. In [4], Candan studied at nonoscillatory solutions of higher order differential and delay differential equations with forcing term. Furthermore, as can be viewed in references [5-13], many researchers have conducted studies on the non-oscillatory solutions of differential equations and oscillation theory.

The aim of this research is to provide adequate expressions for non-oscillatory solutions of (1) based on various cases of the intervality of p(t).

Let $T_0 = \min\left\{t_1 - \tau, \inf_{t\geq t_1} \min_{\xi\in[a_1,b_1]} \sigma_1(t,\xi), \ \inf_{t\geq t_1} \min_{\xi\in[a_2,b_2]} \sigma_2(t,\xi)\right\}$. We define a solution of Equation (1) as a function $x \in C([T_0,\infty),\mathbb{R})$ that satisfies $x(t) - p(t)x(t-\tau)$ being twice continuously differentiable on $[t_1, \infty)$ and also fulfills Equation (1) for $t \geq t_1$.

Consistent with established convention, a solution to Equation (1) is deemed oscillatory if it contains zeros of arbitrarily large magnitude. Conversely, if this criterion is not met, the solution is classified as non-oscillatory. The forthcoming theorem will be employed to substantiate the subsequent theorems.

2. MAIN RESULTS

Theorem 2.1. Provided that (3) - (5) are satisfies along with $0 \leq p(t) \leq p < 1$, (1) admits a bounded nonoscillatory solution.

Proof. We consider *X* be the set representing all functions that are both continuous and bounded on a specific interval $[t_0, \infty)$, characterized by the supremum norm. *X* being a Banach space is self-evident. If condition (4) holds for $d > 0$, the case of $d < 0$ can be approached in a similar manner. Let $N_2 = d$ and take N_1 and N_2 are positive constants satisfiying $N_1 < (1-p)\,N_2$. We denote a closed, bounded, and convex subset A of *X* by

$$A = \{x \in X : N_1 \leq x(t) \leq N_2, \qquad t \geq t_0\}.$$

With reference to Equations (3) through (5), we are able to identify a suitably large $t_1 > t_0$ such that for $t \geq t_1$ and $\xi \in [a_i, b_i]$ with $i = 1, 2$, the conditions $t - \tau \geq t_0$, $\sigma_i(t,\xi) \geq t_0$ hold and

$$p + \int_{t_1}^{\infty} s[(b_1 - a_1)q_1(s) + (b_2 - a_2)q_2(s)]\,ds \leq \theta_1 < 1, \tag{8}$$

$$\int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s,d) + |g(s)|]\,ds \le \alpha - N_1, \tag{9}$$

$$\int_{t_1}^{\infty} s[(b_2 - a_2)f_2(s,d) + |g(s)|]\,ds \le (1-p)N_2 - \alpha, \tag{10}$$

where θ_1 is a constant and $\alpha \in (N_1, (1-p)N_2)$.

Consider the mapping $S : A \longrightarrow X$, which has the following definition:

$$(Sx)(t) = \begin{cases} \alpha + p(t)x(t-\tau) - \int_t^{\infty}(s-t)\left[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s,\xi)))d\xi \right. \\ \left. \quad - \int_{a_2}^{b_2} f_2(s, x(\sigma_2(s,\xi)))d\xi - g(s)\right] ds, & t \ge t_1 \\ (Sx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

It can be rigorously demonstrated that the mapping *Sx* maintains continuity across its entire domain. For every $x \in A$ and $t \ge t_1$, drawing upon Equation (10), we can infer the following

$$\begin{aligned} (Sx)(t) &= \alpha + p(t)x(t-\tau) - \int_t^{\infty}(s-t)\left[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s,\xi)))d\xi \right. \\ &\quad \left. - \int_{a_2}^{b_2} f_2(s, x(\sigma_2(s,\xi)))d\xi - g(s)\right] ds \\ &\le \alpha + pN_2 + \int_{t_1}^{\infty} s[(b_2 - a_2)f_2(s,d) + |g(s)|]\,ds \\ &\le N_2 \end{aligned}$$

and in light of Equation (9), we derive

$$\begin{aligned} (Sx)(t) &= \alpha + p(t)x(t-\tau) - \int_t^{\infty}(s-t)\left[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s,\xi)))d\xi \right. \\ &\quad \left. - \int_{a_2}^{b_2} f_2(s, x(\sigma_2(s,\xi)))d\xi - g(s)\right] ds \\ &\ge \alpha - \int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s,d) + |g(s)|]\,ds \\ &\ge N_1. \end{aligned}$$

We have hence demonstrated that $SA \subset A$. Next, we are going to present a proof that *S* is a mapping that contracts onto set A. In particular, when considering $x, y \in A$ and $t \ge t_1$, it becomes apparent from (2) and (8) that

$$\begin{aligned} |(Sx)(t) - (Sy)(t)| &\le p|x(t-\tau) - y(t-\tau)| \\ &\quad + \int_t^{\infty}(s-t)\int_{a_1}^{b_1}|f_1(s, x(\sigma_1(s,\xi))) - f_1(s, y(\sigma_1(s,\xi)))|d\xi ds \\ &\quad + \int_t^{\infty}(s-t)\int_{a_2}^{b_2}|f_2(s, x(\sigma_2(s,\xi))) - f_2(s, y(\sigma_2(s,\xi)))|d\xi ds \\ &\le p|x(t-\tau) - y(t-\tau)| \\ &\quad + \int_{t_1}^{\infty} s \int_{a_1}^{b_1} q_1(s)\left|x\left(\sigma_1(s,\xi)\right) - y\left(\sigma_1(s,\xi)\right)\right|d\xi ds \\ &\quad + \int_{t_1}^{\infty} s \int_{a_1}^{b_1} q_2(s)|x(\sigma_2(s,\xi)) - y(\sigma_2(s,\xi))|d\xi ds \\ &\le \|x-y\|\left(p + \int_{t_1}^{\infty} s[(b_1 - a_1)q_1(s) + (b_2 - a_2)q_2(s)\,]\,ds\right) \\ &\le \theta_1\|x-y\|. \end{aligned}$$

Based on the supremum norm, we conclude that

$$\|Sx - Sy\| \le \theta_1\|x-y\|.$$

Due to the condition $\theta_1 < 1$, it follows that S is a mapping that contracts onto set A. Applying the Banach Contraction Mapping Principle, we can establish the presence and uniqueness of a fixed point $x \in A$ such that $Sx = x$. This fixed point serves as a positive solution to Equation (1), demonstrating the effectiveness of the principle in guaranteeing the existence of solutions within the specified context. Consequently, we have established the completion of the proof.

Theorem 2.2. Provided that (3)- (5) are satisfies along with $0 < p_1 \leq p(t) \leq p_2 < \infty$, (1) admits a bounded nonoscillatory solution.

Proof. We consider X be the set representing all functions that are both continuous and bounded on a specific interval $[t_0, \infty)$, characterized by the supremum norm. X being a Banach space is self-evident. If condition (4) holds for $d > 0$, the case of $d < 0$ can be approached in a similar manner. Let $N_4 = d$ and take N_3 and N_4 are positive constants satisfiying $p_2 N_3 < (p_1 - 1)\, N_4$. We denote a closed, bounded, and convex subset A of X by

$$A = \{x \in X : N_3 \leq x(t) \leq N_4, \quad t \geq t_0\}.$$

Upon consideration of (3) through (5) we are able to identify a suitably large $t_1 > t_0$ such that $\sigma_i(t+\tau, \xi) \geq t_0$, $\xi \in [a_i, b_i]$ with $i = 1, 2$ also for $t \geq t_1$ and $\alpha \in (p_2 N_3, (p_1 - 1)N_4)$ there exist a constant θ_2 such that

$$\frac{1}{p_1}\left[1 + \int_{t_1}^{\infty} s[(b_1 - a_1)q_1(s) + (b_2 - a_2)q_2(s)]ds\right] \leq \theta_2 < 1, \tag{11}$$

$$\int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s, d) + |g(s)|]\, ds \leq (p_1 - 1)N_4 - \alpha, \tag{12}$$

$$\int_{t_1}^{\infty} s[(b_2 - a_2)f_2(s, d) + |g(s)|]\, ds \leq \alpha - p_2 N_3. \tag{13}$$

Now, consider the mapping $S : A \longrightarrow X$, which has the following definition:

$$(Sx)(t) = \begin{cases} \dfrac{1}{p(t+\tau)}\left[\alpha + x(t+\tau) + \displaystyle\int_{t+\tau}^{\infty} (s - (t+\tau))\left[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s, \xi)))d\xi \right.\right. \\ \qquad \left.\left. - \displaystyle\int_{a_2}^{b_2} f_2(s, x(\sigma_2(s, \xi)))d\xi - g(s)\right] ds\right], \qquad t \geq t_1 \\ (Sx)(t_1), \qquad\qquad t_0 \leq t \leq t_1. \end{cases}$$

It can be firmly proven that the mapping Sx maintains continuity across its entire domain. For every $x \in A$ and $t \geq t_1$, drawing upon Equation (12), we can conclude the following

$$\begin{aligned} (Sx)(t) &= \frac{1}{p(t+\tau)}\left[\alpha + x(t+\tau) + \int_{t+\tau}^{\infty} (s - (t+\tau))\left[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s, \xi)))d\xi \right.\right. \\ &\quad \left.\left. - \int_{a_2}^{b_2} f_2(s, x(\sigma_2(s, \xi)))d\xi - g(s)\right] ds\right] \\ &\leq \frac{1}{p_1}\left[\alpha + N_4 + \int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s, d) + |g(s)|]ds\right] \\ &\leq N_4 \end{aligned}$$

furthermore, by using Equation (13) along with the previous analysis, we arrive at

$$\begin{aligned} (Sx)(t) &= \frac{1}{p(t+\tau)}\left[\alpha + x(t+\tau) + \int_{t+\tau}^{\infty} (s - (t+\tau))\left[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s, \xi)))d\xi \right.\right. \\ &\quad \left.\left. - \int_{a_2}^{b_2} f_2(s, x(\sigma_2(s, \xi)))d\xi - g(s)\right] ds\right] \end{aligned}$$

$$\geq \frac{1}{p_2}\left[\alpha - \int_{t_1}^{\infty} s[(b_2 - a_2)f_2(s,d) + |g(s)|]ds\right]$$
$$\geq N_3.$$

We have hence demonstrated that $SA \subset A$. Next, we are going to present a proof that *S* is a mapping that contracts onto set A. In particular, when considering $x, y \in A$ and $t \geq t_1$, it becomes apparent from (2) and (11) that

$$\begin{aligned}|(Sx)(t) - (Sy)(t)| \leq & \frac{1}{p(t+\tau)}[|x(t+\tau) - y(t+\tau)| \\ & + \int_{t+\tau}^{\infty}(s-(t+\tau))\int_{a_1}^{b_1}|f_1(s,x(\sigma_1(s,\xi))) - f_1(s,y(\sigma_1(s,\xi)))|d\xi ds \\ & + \int_{t+\tau}^{\infty}(s-(t+\tau))\int_{a_2}^{b_2}|f_2(s,x(\sigma_2(s,\xi))) - f_2(s,y(\sigma_2(s,\xi)))|d\xi ds\Big] \\ & \leq \frac{1}{p_1}[|x(t+\tau) - y(t+\tau)| \\ & + \int_{t_1}^{\infty} s\int_{a_1}^{b_1} q_1(s)\big|x\big(\sigma_1(s,\xi)\big) - y\big(\sigma_1(s,\xi)\big)\big|d\xi ds \\ & + \int_{t_1}^{\infty} s\int_{a_1}^{b_1} q_2(s)\big|x\big(\sigma_2(s,\xi)\big) - y\big(\sigma_2(s,\xi)\big)\big|d\xi ds\Big] \\ & \leq \frac{\|x-y\|}{p_1}\left[1 + \int_{t_1}^{\infty} s[(b_1 - a_1)q_1(s) + (b_2 - a_2)q_2(s)]ds\right] \\ & \leq \theta_2\|x - y\|.\end{aligned}$$

Based on the supremum norm, we conclude that

$$\|Sx - Sy\| \leq \theta_2\|x - y\|.$$

Due to the condition $\theta_2 < 1$, *S* contracts onto set A. Applying the Banach Contraction Mapping Principle, we can establish the presence and uniqueness of a fixed point $x \in A$ such that $Sx = x$. This fixed point serves as a positive solution to Equation (1), demonstrating the effectiveness of the principle in guaranteeing the presence of solutions within the specified context. Consequently, we have established the completion of the proof.

Theorem 2.3. Provided that (3) - (5) are satisfies along with $-1 < -p \leq p(t) \leq 0$, (1) admits a bounded nonoscillatory solution.

Proof. We consider *X* be the set representing all functions that are both continuous and bounded on a specific interval $[t_0, \infty)$, characterized by the supremum norm. *X* being a Banach space is self-evident. If condition (4) holds for $d > 0$, the case of $d < 0$ can be approached in a similar manner. Let $N_6 = d$ and take N_5 and N_6 are positive constants satisfiying $N_5 + pN_6 < N_6$. We denote a closed, bounded, and convex subset A of *X* by

$$A = \{x \in X : N_5 \leq x(t) \leq N_6, \quad t \geq t_0\}.$$

With reference to Equations (3) through (5), we are able to identify a suitably large $t_1 > t_0$ such that $t - \tau \geq t_0$, $\sigma_i(t+\tau,\xi) \geq t_0$, $\xi \in [a_i, b_i]$ with $i = 1, 2$ also for $t \geq t_1$ there exist a constant θ_3 and $\alpha \in (N_5 + p N_6, N_6)$ such that

$$p + \int_{t_1}^{\infty} s[(b_1 - a_1)q_1(s) + (b_2 - a_2)q_2(s)]\,ds \leq \theta_3 < 1, \tag{14}$$

$$\int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s,d) + |g(s)|]\,ds \leq \alpha - N_5 - pN_6 \tag{15}$$

$$\int_{t_1}^{\infty} s[(b_2 - a_2)f_2(s,d) + |g(s)|]\,ds \leq N_6 - \alpha. \tag{16}$$

Consider the mapping $S : A \longrightarrow X$, which has the following definition:

$$(Sx)(t) = \begin{cases} \alpha + p(t)x(t-\tau) - \int_t^{\infty}(s-t)\left[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi \right. \\ \left. \quad - \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\right]ds, \qquad t \geq t_1 \\ (Sx)(t_1), \qquad\qquad t_0 \leq t \leq t_1. \end{cases}$$

It can be firmly proven that the mapping *Sx* maintains continuity across its entire domain. For every $x \in A$ and $t \geq t_1$, drawing upon Equation (16), we can conclude the following

$$\begin{aligned}(Sx)(t) = {} & \alpha + p(t)x(t-\tau) - \int_t^{\infty}(s-t)\left[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi \right. \\ & \left. - \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\right]ds \\ & \leq \alpha + \int_{t_1}^{\infty} s[(b_2-a_2)f_2(s,d) + |g(s)|]\,ds \\ & \leq N_6\end{aligned}$$

and in consideration of (15), our result is

$$\begin{aligned}(Sx)(t) = {} & \alpha + p(t)x(t-\tau) - \int_t^{\infty}(s-t)\left[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi \right. \\ & \left. - \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\right]ds \\ & \geq \alpha - pN_6 - \int_{t_1}^{\infty} s[(b_1-a_1)f_1(s,d) + |g(s)|]\,ds \\ & \geq N_5.\end{aligned}$$

So, we have demonstrated that $SA \subset A$. Next, we are going to present a proof that *S* is a mapping that contracts onto set A. In particular, when considering $x, y \in A$ and $t \geq t_1$, it becomes apparent from (2) and (14) that

$$\begin{aligned}|(Sx)(t) - (Sy)(t)| \leq {} & p|x(t-\tau) - y(t-\tau)| \\ & + \int_t^{\infty}(s-t)\int_{a_1}^{b_1}|f_1(s,x(\sigma_1(s,\xi))) - f_1(s,y(\sigma_1(s,\xi)))|d\xi ds \\ & + \int_t^{\infty}(s-t)\int_{a_2}^{b_2}|f_2(s,x(\sigma_2(s,\xi))) - f_2(s,y(\sigma_2(s,\xi)))|d\xi ds \\ & \leq p|x(t-\tau) - y(t-\tau)| \\ & + \int_{t_1}^{\infty} s\int_{a_1}^{b_1} q_1(s)\left|x\left(\sigma_1(s,\xi)\right) - y\left(\sigma_1(s,\xi)\right)\right|d\xi ds \\ & + \int_{t_1}^{\infty} s\int_{a_1}^{b_1} q_2(s)\left|x\left(\sigma_2(s,\xi)\right) - y\left(\sigma_2(s,\xi)\right)\right|d\xi ds \\ & \leq \|x-y\|\left(p + \int_{t_1}^{\infty} s[(b_1-a_1)q_1(s) + (b_2-a_2)q_2(s)\,]\,ds\right) \\ & \leq \theta_3\|x-y\|.\end{aligned}$$

Based on the supremum norm, we conclude that

$$\|Sx - Sy\| \leq \theta_3\|x-y\|.$$

Due to the condition $\theta_3 < 1$, *S* contracts onto set A. Applying the Banach Contraction Mapping Principle, we can establish the presence and uniqueness of a fixed point $x \in A$ such that $Sx = x$. This fixed point serves as a positive solution to Equation (1), demonstrating the effectiveness of the principle in guaranteeing the existence of solutions within the specified context. Consequently, we have established the completion of the proof.

Theorem 2.4. Provided that (3) - (5) are satisfies along with $-\infty < -p_1 \leq p(t) \leq -p_2 < -1$, Equation (1) admits a bounded nonoscillatory solution.

Proof. We consider X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. X being a Banach space is self-evident. If condition (4) holds for $d > 0$, the case of $d < 0$ can be approached in a similar manner. Let $N_8 = d$ and take N_7 and N_8 are positive constants satisfiying $p_1 N_7 + N_8 < p_2 N_8$. We denote a closed, bounded, and convex subset A of X by

$$A = \{x \in X : N_7 \leq x(t) \leq N_8, \quad t \geq t_0\}.$$

From the examination of (3) to (5), we note the existence of a sufficiently large $t_1 > t_0$ such that $\sigma_i(t+\tau, \xi) \geq t_0$, $\xi \in [a_i, b_i]$ where $i = 1, 2$ also for $t \geq t_1$ there exist a constant θ_4 and $\alpha \in (p_1 N_7 + N_8, p_2 N_8)$ such that

$$\frac{1}{p_2}\left[1 + \int_{t_1}^{\infty} s[(b_1 - a_1)q_1(s) + (b_2 - a_2)q_2(s)]ds\right] \leq \theta_4 < 1, \tag{17}$$

$$\int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s,d) + |g(s)|]\, ds \leq \alpha - p_1 N_7 - N_8, \tag{18}$$

$$\int_{t_1}^{\infty} s[(b_2 - a_2)f_2(s,d) + |g(s)|]\, ds \leq p_2 N_8 - \alpha\,. \tag{19}$$

Let S denote a mapping $S : A \longrightarrow X$, which is defined in the following way:

$$(Sx)(t) = \begin{cases} -\dfrac{1}{p(t+\tau)}\left[\alpha - x(t+\tau) - \displaystyle\int_{t+\tau}^{\infty}(s-(t+\tau))\left[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s,\xi)))d\xi \right.\right. \\ \qquad \left.\left. - \displaystyle\int_{a_2}^{b_2} f_2(s, x(\sigma_2(s,\xi)))d\xi - g(s)\right]ds\right], \qquad t \geq t_1 \\ (Sx)(t_1), \qquad\qquad t_0 \leq t \leq t_1. \end{cases}$$

The continuity of the function Sx is readily apparent. Furthermore, considering for every $x \in A$ and $t \geq t_1$, from Equation (19), we get

$$\begin{aligned} (Sx)(t) &= -\tfrac{1}{p(t+\tau)}\Big[\alpha - x(t+\tau) - \textstyle\int_{t+\tau}^{\infty}(s-(t+\tau))\Big[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s,\xi)))d\xi \\ &\quad - \textstyle\int_{a_2}^{b_2} f_2(s, x(\sigma_2(s,\xi)))d\xi - g(s)\Big]ds\Big] \\ &\leq \tfrac{1}{p_2}\left[\alpha + \textstyle\int_{t_1}^{\infty} s[(b_2 - a_2)f_2(s,d) + |g(s)|]ds\right] \\ &\leq N_8 \end{aligned}$$

and by putting (18) into context, we get the following

$$\begin{aligned} (Sx)(t) &= -\tfrac{1}{p(t+\tau)}\Big[\alpha - x(t+\tau) - \textstyle\int_{t+\tau}^{\infty}(s-(t+\tau))\Big[\int_{a_1}^{b_1} f_1(s, x(\sigma_1(s,\xi)))d\xi \\ &\quad - \textstyle\int_{a_2}^{b_2} f_2(s, x(\sigma_2(s,\xi)))d\xi - g(s)\Big]ds\Big] \\ &\geq -\tfrac{1}{p(t+\tau)}\left[\alpha - N_8 - \textstyle\int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s,d) + |g(s)|]ds\right] \\ &\geq -\tfrac{1}{p_1}\left[\alpha - N_8 - \textstyle\int_{t_1}^{\infty} s[(b_1 - a_1)f_1(s,d) + |g(s)|]ds\right] \\ &\geq N_7. \end{aligned}$$

So, we have demonstrated that $SA \subset A$. Next, we are going to present a proof that S is a mapping that contracts onto set A. In particular, when considering $x, y \in A$ and $t \geq t_1$, it becomes apparent from (2) and (17) that

$$\begin{aligned}|(Sx)(t)-(Sy)(t)| &\leq \tfrac{1}{|p(t+\tau)|}[|x(t+\tau)-y(t+\tau)| \\ &+ \int_{t+\tau}^{\infty}(s-(t+\tau))\int_{a_1}^{b_1}|f_1(s,x(\sigma_1(s,\xi)))-f_1(s,y(\sigma_1(s,\xi)))|d\xi ds \\ &+ \int_{t+\tau}^{\infty}(s-(t+\tau))\int_{a_2}^{b_2}|f_2(s,x(\sigma_2(s,\xi)))-f_2(s,y(\sigma_2(s,\xi)))|d\xi ds\Big] \\ &\leq \frac{1}{p_2}[|x(t+\tau)-y(t+\tau)| \\ &+ \int_{t_1}^{\infty} s\int_{a_1}^{b_1} q_1(s)\big|x\big(\sigma_1(s,\xi)\big)-y\big(\sigma_1(s,\xi)\big)\big|d\xi ds \\ &+ \int_{t_1}^{\infty} s\int_{a_1}^{b_1} q_2(s)\big|x\big(\sigma_2(s,\xi)\big)-y\big(\sigma_2(s,\xi)\big)\big|d\xi ds\Big] \\ &\leq \frac{\|x-y\|}{p_2}\Big[1+\int_{t_1}^{\infty} s[(b_1-a_1)q_1(s)+(b_2-a_2)q_2(s)]ds\Big] \\ &\leq \theta_4\|x-y\|.\end{aligned}$$

Based on the supremum norm, we conclude that

$$\|Sx-Sy\| \leq \theta_4\|x-y\|.$$

Due to the condition $\theta_4 < 1$, S contracts onto set A. Applying the Banach Contraction Mapping Principle, we can establish the presence and uniqueness of a fixed point $x \in A$ such that $Sx = x$. This fixed point serves as a positive solution to Equation (1), demonstrating the effectiveness of the principle in guaranteeing the existence of solutions within the specified context. Consequently, we have established the completion of the proof.

Example 2.3. Taking into account $t > 0$ investigate the following

$$\begin{aligned}&(x(t)-\exp(-t)x(t-1))'' + \int_0^1 \exp(-t-1)x(t-2\xi)d\xi - \tfrac{1}{2}\int_1^3 \exp(-t-2)x(t-\xi)d\xi \\ &= -4\exp(1-2t)+\exp(-t-1)-\exp(-t-2). \qquad (20)\end{aligned}$$

Notice that $p(t) = \exp(-t)$, $\tau = 1$, $\sigma_1(t,\xi) = t-2\xi$, $\sigma_2(t,\xi) = t-\xi$, $f_1(t,u) = \exp(-t-1)u$, $f_2(t,u) = \frac{\exp(-t-2)u}{2}$ and $g(t) = -4\exp(1-2t)+\exp(-t-1)-\exp(-t-2)$. It is evident that the function serves as a nonoscillatory solution to Equation (20)." We can confirm that the prerequisites outlined in Theorem 2.1 are fully met. It is worth mentioning that $x(t) = \exp(-t)+1$ is a nonoscillatory solution of (20).

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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