T_2 AND T_3 OBJECTS AT p IN THE CATEGORY OF PROXIMITY SPACES

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Abstract. In previous papers, various notions of pre-Hausdorff, Hausdorff and regular objects at a point p in a topological category were introduced and compared. The main objective of this paper is to characterize each of these notions of pre-Hausdorff, Hausdorff and regular objects locally in the category of proximity spaces. Furthermore, the relationships that arise among the various PreT_2, T_i, $i = 0, 1, 2, 3$, structures at a point p are investigated. Finally, we examine the relationships between the generalized separation properties and the separation properties at a point p in this category.

Keywords: topological category; proximity space; Hausdorff space; regular space

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1. Introduction

Proximity spaces were discovered by Efremovich during the first part of 1930s and later axiomatized (see [15], [16]). He characterized the proximity relation "A is near B" for subsets A and B of any set X. This theory was improved by Smirnov (see [26]). He showed which topological spaces admit a proximity relation compatible with the given topology, and he was also the first to discover the relationship between proximities and uniformities. The most extensive work on the theory of proximity spaces was done by Naimpally and Warrack (see [23]). All our preliminary information on proximity spaces can be found in this source.

In 1991, Baran in [2] introduced separation properties for an arbitrary topological category over sets. He in [2] defined these generalizations first at a point p, i.e. locally (see [4], [20]), then he generalized this to point free definitions by using the generic element, the method of topos theory (see [19], page 39). Using local separation properties, Baran in [2], [3] introduced the notion of strong closedness in set-based

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topological categories which forms closure operators in sense of Dikranjan and Giuli (see [14]) in some well-known topological categories. He used the (strong) closed objects to generalize each of the notions of connectedness (see [12]), Hausdorffness (see [2], [11]), compactness and perfectness (see [6]) to arbitrary set-based topological categories.

The main goal of this paper is to give the characterization of each various notions of local pre-Hausdorff, local Hausdorff and local regular proximity spaces as well as to investigate how these notions are related, and compare the generalized separation properties and the local separation properties in the category of proximity spaces.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U}\colon\ \mathcal{E} \to \mathcal{B}$ is said to be topological or $\mathcal{E}$ is said to be a topological category over $\mathcal{B}$, if $\mathcal{U}$ is concrete (i.e. faithful and amnestic), has small fibers, and every $\mathcal{U}$-source has an initial lift or, equivalently, each $\mathcal{U}$-sink has a final lift (see [1]).

Recall in [1] or [24] that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \mathrm{Ob}(\mathcal{E})$), a topological category, is discrete if and only if every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$.

Definition 2.1 ([23])**.** An Efremovich proximity (EF-proximity) space is a pair (X, δ), where X is a set and δ is a binary relation on the power set of X such that

(P1) $A\,\delta\,B$ if and only if $B\,\delta\,A$;
(P2) $A\,\delta\,(B \cup C)$ if and only if $A\,\delta\,B$ or $A\,\delta\,C$;
(P3) $A\,\delta\,B$ implies $A, B \neq \emptyset$;
(P4) $A \cap B \neq \emptyset$ implies $A\,\delta\,B$;
(P5) $A\,\overline{\delta}\,B$ implies there is an $E \subseteq X$ such that $A\,\overline{\delta}\,E$ and $(X - E)\,\overline{\delta}\,B$,

where $A\,\overline{\delta}\,B$ means that it is not true that $A\,\delta\,B$.

A function $f\colon\ (X, \delta) \to (Y, \delta')$ between two proximity spaces is called a *proximity mapping* (or a *p-map*) if and only if $f(A)\,\delta'\,f(B)$ whenever $A\,\delta\,B$. It can easily be shown that f is a p-map if and only if for subsets C and D of Y, $f^{-1}(C)\,\overline{\delta}\,f^{-1}(D)$ whenever $C\,\overline{\delta}'\,D$.

In a proximity space (X, δ), we write $A \ll B$ if and only if $A\,\overline{\delta}\,(X - B)$. The relation $\ll$ is called p-neighborhood relation or the strong inclusion. When $A \ll B$, we say that B is a p-neighborhood of A or A is strongly contained in B, see [17] or [23].

We denote the category of proximity spaces and proximity mappings by Prox. Hunsaker and Sharma in [18] showed that the forgetful functor $\mathcal{U}\colon$ Prox $\to$ Set is topological.

Definition 2.2. Let $\mathfrak{B}$ be a proximity-base on a set X and let a binary relation δ on $P(X)$ be defined as follows: $(A, B) \in \delta$ if, given any finite covers $\{A_i\colon\ 1 \leqslant i \leqslant n\}$ and $\{B_j\colon\ 1 \leqslant j \leqslant m\}$ of A and B, respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$; δ is a proximity on X finer than the relation $\mathfrak{B}$, see [18] or [25].

Definition 2.3. Let X be a nonempty set, (X_i, δ_i), $i \in I$ be a family of proximity spaces and $f_i\colon\ X \to X_i$ be a source in Set. Define a binary relation $\mathfrak{B}$ on $P(X)$ as follows: for $A, B \in P(X)$, $A\ \mathfrak{B}\ B$ if and only if $f_i(A)\ \delta_i\ f_i(B)$ for all $i \in I$. $\mathfrak{B}$ is a proximity-base on X (see [25], Theorem 3.8). The initial proximity structure δ on X generated by the proximity base $\mathfrak{B}$ is given for $A, B \in P(X)$ as follows: $A\ \delta\ B$ if and only if for any finite covers $\{A_i\colon\ 1 \leqslant i \leqslant n\}$ and $\{B_j\colon\ 1 \leqslant j \leqslant m\}$ of A and B, respectively, there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$, see [25].

Definition 2.4. Let (X, δ) be a proximity space, Y a nonempty set and f be a function from a proximity space (X, δ) onto a set Y. The strong inclusion $\ll^*$ induced by the finest proximity δ^* (the quotient proximity) on Y making f proximally continuous is given for every $A, B \subset Y$ as follows: $A \ll^* B$ if and only if for each binary rational s in $[0, 1]$ there is a $C_s \subset Y$ such that $C_0 = A$, $C_1 = B$ and $s < t$ implies $f^{-1}(C_s) \ll_\delta f^{-1}(C_t)$ (see [17] or [27], page 276), where $\ll_\delta$ represents the strong inclusion induced by the proximity δ on X. In addition, if $f\colon\ (X, \delta) \to (X, \delta^*)$ is a one-to-one p-quotient map, then $A\ \delta^*\ B$ if and only if $f^{-1}(A)\ \delta\ f^{-1}(B)$, see [17], page 591.

Definition 2.5. Let X be set and $p \in X$. Let $X \vee_p X$ be the wedge at p (see [2]), i.e. two disjoint copies of X identified at p, i.e. the pushout of $p\colon\ 1 \to X$ along itself (where 1 is the terminal object in Set). An epi sink $\{i_1, i_2\colon\ (X, \delta) \to (X \vee_p X, \delta')\}$ (p-maps), where i_1, i_2 are the canonical injections, in Prox is a final lift if and only if the following statement holds. For each pair A, B in the different component of $X \vee_p X$, $A\ \delta'\ B$ if and only if there exist sets C, D in X such that $C\ \delta\ \{p\}$ and $\{p\}\ \delta\ D$ with $i_k^{-1}(A) = C$ and $i_j^{-1}(B) = D$ for $k, j = 1, 2$ and $k \neq j$. If A and B are in the same component of the wedge, then $A\ \delta'\ B$ if and only if there exist sets C, D in X such that $C\ \delta\ D$ and $i_k^{-1}(A) = C$ and $i_k^{-1}(B) = D$ for some $k = 1, 2$. Specially, if $i_k(C) = A$ and $i_k(D) = B$, then $(i_k(C), i_k(D)) \in \delta'$ if and only if $(i_k^{-1}(i_k(C)), i_k^{-1}(i_k(D))) = (C, D) \in \delta$.

Definition 2.6 ([23], page 9)**.** Let X be a nonempty set. The discrete proximity structure δ on X is defined as follows for $A, B \subset X\colon\ A\ \delta\ B$ if and only if $A \cap B \neq \emptyset$.

3. PreT_2 and T_2 proximity spaces at a point

In this section, we give the characterization of Pre$\overline{T}_2$, PreT_2', $\overline{T}_2$ and T_2' proximity spaces at a point p.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal p-axis map $A_p\colon B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map $S_p\colon B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\bigtriangledown_p\colon B \vee_p B \to B$ is given by $\bigtriangledown_p(x_i) = x$ for $i = 1, 2$ (see [2], [3]).

Note that the maps A_p, S_p and $\bigtriangledown_p$ are the unique maps arising from the above pushout diagram for which $A_p i_1 = (\mathrm{id}, f)$, $S_p i_1 = (\mathrm{id}, \mathrm{id})\colon B \to B^2$, $A_p i_2 = S_p i_2 = (f, \mathrm{id})\colon B \to B^2$, and $\bigtriangledown_p i_j = \mathrm{id}$, $j = 1, 2$, respectively, where $\mathrm{id}\colon B \to B$ is the identity map and $f\colon B \to B$ is the constant map at p (see [8]).

R e m a r k 3.1. We define p_1, p_2 by $1 + f$, $f + 1\colon B \vee_p B \to B$, respectively, where $1\colon B \to B$ is the identity map, $f\colon B \to B$ is a constant map at p (i.e. having value p). Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, $\pi_2 S_p = \bigtriangledown_p$, where $\pi_i\colon B^2 \to B$ is the ith projection, $i = 1, 2$. When showing that A_p and S_p are initial, it is sufficient to show that (p_1 and p_2) and (p_1 and $\bigtriangledown_p$) are initial lifts, respectively, see [2], [3].

Definition 3.1 (cf. [2], [3])**.** Let $\mathcal{U}\colon \mathcal{E} \to \mathrm{Set}$ be a topological functor, X an object in $\mathcal{E}$, p a point in $\mathcal{U}(X) = B$.

(1) X is $\overline{T}_0$ at p if and only if the initial lift of the $\mathcal{U}$-source $\{A_p\colon B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\bigtriangledown_p\colon B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.

(2) X is T_0' at p if and only if the initial lift of the $\mathcal{U}$-source $\{\mathrm{id}\colon B \vee_p B \to \mathcal{U}(X \vee_p X) = B \vee_p B$ and $\bigtriangledown_p\colon B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $X \vee_p X$ is the wedge in $\mathcal{E}$, i.e. the final lift of the $\mathcal{U}$-sink $\{i_1, i_2\colon \mathcal{U}(X) = B \to B \vee_p B\}$, where i_1, i_2 denote the canonical injections.

(3) X is T_1 at p if and only if the initial lift of the $\mathcal{U}$-source $\{S_p\colon B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\bigtriangledown_p\colon B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.

(4) X is Pre$\overline{T}_2$ at p if and only if the initial lift of the $\mathcal{U}$-source $S_p\colon B \vee_p B \to \mathcal{U}(X^2) = B^2$ and the initial lift of the $\mathcal{U}$-source $A_p\colon B \vee_p B \to \mathcal{U}(X^2) = B^2$ agree.

(5) X is PreT_2' at p if and only if the initial structure on $B \vee_p B$ induced from $\mathcal{U}$-source $S_p\colon B \vee_p B \to \mathcal{U}(X^2) = B^2$ and the final structure on $B \vee_p B$ induced from $\mathcal{U}$-sink $\{i_1, i_2\colon \mathcal{U}(X) = B \to B \vee_p B\}$ coincide.

(6) X is $\overline{T}_2$ at p if and only if X is $\overline{T}_0$ at p and Pre$\overline{T}_2$ at p.

(7) X is T_2' at p if and only if X is T_0' at p and $\mathrm{Pre}\,T_2'$ at p.

R e m a r k 3.2. Note that for the category Top of topological spaces we have:

(1) A topological space X is $\mathrm{Pre}\,\overline{T}_2$ at p if and only if X is $\mathrm{Pre}\,T_2'$ at p if and only if for each point x distinct from p with $x, p \in X$, if the subspace of X is not indiscrete, then there exist disjoint neighborhoods of x and p, see [2], [5].
(2) A topological space X is $\overline{T}_2$ at p if and only if X is T_2' at p if and only if for each point x distinct from p there exist disjoint neighborhoods of x and p, see [2], [5].

The following result is given in [21].

Theorem 3.1. *Let (X, δ) be an Efremovich proximity space and $p \in X$.*

(1) *(X, δ) is T_1 at p or $\overline{T}_0$ at p if and only if for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$.*
(2) *(X, δ) is T_0' at p for every $p \in X$.*

Theorem 3.2. *An Efremovich proximity space is $\mathrm{Pre}\,\overline{T}_2$ at p for every $p \in X$.*

P r o o f. Let (X, δ) be any Efremovich proximity space. By Definitions 2.3, 3.1 and Remark 3.1, we will show that (X, δ) is $\mathrm{Pre}\,\overline{T}_2$ at p, i.e. for any pair U and V in the wedge, $\pi_1 A_p(U)\,\delta\,\pi_1 A_p(V)$, $\pi_2 A_p(U)\,\delta\,\pi_2 A_p(V)$ if and only if $\pi_1 S_p(U)\,\delta\,\pi_1 S_p(V)$, $\pi_2 S_p(U)\,\delta\,\pi_2 S_p(V)$, respectively.

We consider various possibilities for U and V; namely $\{x_1\} \subseteq U$, $\{x_2\} \subseteq U$ or $\{x_1, x_2\} \subseteq U$ and $\{x_1\} \subseteq V$, $\{x_2\} \subseteq V$ or $\{x_1, x_2\} \subseteq V$. By condition (P2) of Definition 2.1 it is sufficient to take "equality" instead of "subset" for the possibilities above.

Consider the case $U = \{x_1\}$ and $V = \{x_1\}$: if $\pi_1 A_p(\{x_1\})\,\delta\,\pi_1 A_p(\{x_1\}) = \{x\}\,\delta\,\{x\}$ and $\pi_2 A_p(\{x_1\})\ \delta\ \pi_2 A_p(\{x_1\}) = \{p\}\ \delta\ \{p\}$, then $\pi_1 S_p(\{x_1\})\ \delta\ \pi_1 S_p(\{x_1\}) = \{x\}\ \delta\ \{x\} = \pi_2 S_p(\{x_1\})\ \delta\ \pi_2 S_p(\{x_1\})$. Conversely, if $\pi_1 S_p(\{x_1\})\ \delta\ \pi_1 S_p(\{x_1\}) = \{x\}\ \delta\ \{x\} = \pi_2 S_p(\{x_1\})\ \delta\ \pi_2 S_p(\{x_1\})$, then $\pi_1 A_p(\{x_1\})\ \delta\ \pi_1 A_p(\{x_1\}) = \{x\}\ \delta\ \{x\}$ and $\pi_2 A_p(\{x_1\})\,\delta\,\pi_2 A_p(\{x_1\}) = \{p\}\ \delta\ \{p\}$.

Consider the case $U = \{x_1\}$ and $V = \{x_2\}$: if $\pi_1 A_p(\{x_1\})\,\delta\,\pi_1 A_p(\{x_2\}) = \{x\}\,\delta\,\{p\}$ and $\pi_2 A_p(\{x_1\})\ \delta\ \pi_2 A_p(\{x_2\}) = \{p\}\ \delta\ \{x\}$, then $\pi_1 S_p(\{x_1\})\ \delta\ \pi_1 S_p(\{x_2\}) = \{x\}\ \delta\ \{p\}$ and $\pi_2 S_p(\{x_1\})\ \delta\ \pi_2 S_p(\{x_2\}) = \{x\}\ \delta\ \{x\}$ by condition (P4) of Definition 2.1. Conversely, if $\pi_1 S_p(\{x_1\})\ \delta\ \pi_1 S_p(\{x_2\}) = \{x\}\ \delta\ \{p\}$ and $\pi_2 S_p(\{x_1\})\ \delta\ \pi_2 S_p(\{x_2\}) = \{x\}\,\delta\,\{x\}$, then $\pi_1 A_p(\{x_1\})\,\delta\,\pi_1 A_p(\{x_2\}) = \{x\}\,\delta\,\{p\}$ and $\pi_2 A_p(\{x_1\})\,\delta\,\pi_2 A_p(\{x_2\}) = \{p\}\ \delta\ \{x\}$ by condition (P1) of Definition 2.1.

Consider the case $U = \{x_1\}$ and $V = \{x_1, x_2\}$: if $\pi_1 A_p(\{x_1\})\ \delta\ \pi_1 A_p(\{x_1, x_2\}) = \{x\}\ \delta\ \{x, p\}$ and $\pi_2 A_p(\{x_1\})\ \delta\ \pi_2 A_p(\{x_1, x_2\}) = \{p\}\ \delta\ \{p, x\}$, then $\pi_1 S_p(\{x_1\})\ \delta\ \pi_1 S_p(\{x_1, x_2\}) = \{x\}\,\delta\,\{x, p\}$ and $\pi_2 S_p(\{x_1\})\,\delta\,\pi_2 S_p(\{x_1, x_2\}) = \{x\}\,\delta\,\{x\}$ by condition (P4) of Definition 2.1. Conversely, if $\pi_1 S_p(\{x_1\})\,\delta\,\pi_1 S_p(\{x_1, x_2\}) = \{x\}\,\delta\,\{x, p\}$

and $\pi_2 S_p(\{x_1\})\ \delta\ \pi_2 S_p(\{x_1, x_2\}) = \{x\}\ \delta\ \{x\}$, then $\pi_1 A_p(\{x_1\})\ \delta\ \pi_1 A_p(\{x_1, x_2\}) = \{x\}\ \delta\ \{x, p\}$ and $\pi_2 A_p(\{x_1\})\ \delta\ \pi_2 A_p(\{x_1, x_2\}) = \{p\}\ \delta\ \{p, x\}$ by condition (P4) of Definition 2.1.

Similarly, if $\{x_2\} \subseteq U$ or $\{x_1, x_2\} \subseteq U$, and $\{x_1\} \subseteq V$, $\{x_2\} \subseteq V$ or $\{x_1, x_2\} \subseteq V$, then we have $\pi_1 A_p(U) \delta \pi_1 A_p(V)$, $\pi_2 A_p(U) \delta \pi_2 A_p(V)$ if and only if $\pi_1 S_p(U) \delta \pi_1 S_p(V)$, $\pi_2 S_p(U)\ \delta\ \pi_2 S_p(V)$, respectively.

Hence (X, δ) is $\operatorname{Pre} \overline{T}_2$ at p. □

Theorem 3.3. *Let (X, δ) be an Efremovich proximity space and $p \in X$. (X, δ) is $\operatorname{Pre} T_2'$ at p if and only if for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$.*

P r o o f. Suppose (X, δ) is $\operatorname{Pre} T_2'$ at p, i.e. by Definitions 2.3, 2.5, 3.1 and Remark 3.1, for any sets U, V on the wedge, (a) $p_1 U\ \delta\ p_1 V$ and $\bigtriangledown_p U\ \delta\ \bigtriangledown_p V$ if and only if (b) there exists a pair $x, y \in X$ such that $\{x\}\ \delta\ \{y\}$ and $i_k\{x\} = x_k \in U$ and $i_k\{y\} = y_k \in V$ for some $k = 1$ or 2.

For each pair U, V in the different component of $X \vee_p X$, $U\ \delta'\ V$ (δ' is a proximity structure on $X \vee_p X$) if and only if there exist sets C, D in X such that $C\ \delta\ \{p\}$ and $\{p\}\ \delta\ D$ with $i_k^{-1}(U) = C$ and $i_j^{-1}(V) = D$ for $k, j = 1, 2$ and $k \neq j$. If U and V are in the same component of the wedge, then $U\ \delta'\ V$ if and only if there exist sets U, V in X such that $C\ \delta\ D$ and $i_k^{-1}(U) = C$ and $i_k^{-1}(V) = D$ for some $k = 1, 2$. Specially, if $i_k(C) = U$ and $i_k(D) = V$, then $(i_k(C), i_k(D)) \in \delta'$ if and only if $(i_k^{-1}(i_k(C)), i_k^{-1}(i_k(D))) = (C, D) \in \delta$. We shall show that the condition holds.

Suppose for some $x, p \in X$, $(\{x\}, \{p\}) \in \delta$ with $x \neq p$. Then, by Definition 2.3 and Remark 3.1, for $(U, V) \in \delta'$ (δ' is a proximity structure on the wedge) with $U \supseteq \{x_1\}$ and $V \supseteq \{x_2\}$, $p_1 U\ \delta\ p_1 V \supseteq p_1(\{x_1\})\ \delta\ p_1(\{x_2\}) = \pi_1 S_p(\{x_1\})\ \delta\ \pi_1 S_p(\{x_2\}) = \pi_1(\{(x, x)\})\ \delta\ \pi_1(\{(p, x)\}) = \{x\}\ \delta\ \{p\}$, i.e. $(\{x\}, \{p\}) \in \delta$, $\bigtriangledown_p U\ \delta\ \bigtriangledown_p V \supseteq \bigtriangledown_p(\{x_1\})\ \delta\ \bigtriangledown_p(\{x_2\}) = \pi_2 S_p(\{x_1\})\ \delta\ \pi_2 S_p(\{x_2\}) = \pi_2(\{(x, x)\})\ \delta\ \pi_2(\{(p, x)\}) = \{x\}\ \delta\ \{x\}$, i.e. $(\{x\}, \{x\}) \in \delta$, where $\pi_i \colon X^2 \to X$, $i = 1, 2$ are the projection maps.

There exist sets A, B in X such that $A\ \delta\ \{p\}$ and $\{p\}\ \delta\ B$ with $i_k^{-1}(U) = A$ and $i_j^{-1}(V) = B$ for $k, j = 1, 2$ and $k \neq j$. Further $i_k(i_k^{-1}(U)) = i_k(A) \subseteq U$ and $i_j(i_j^{-1}(V)) = i_j(B) \subseteq V$.

If $i_k(A) \subseteq U$ is a subset of the first component of $X \vee_p X$ and $i_k(B) \subseteq V$ is a subset of the second component of $X \vee_p X$, then $\{x_1\} \subseteq i_k(A)$ and $\{x_2\} \subseteq i_k(B)$. But, if $(i_k(A), i_k(B)) \supseteq (\{x_1\}, \{x_2\}) \in \delta'$ for some $(A, B) \in \delta$ and $k = 1$ ($k = 2$), then $(\{x_1\}, \{x_2\}) \subseteq (i_1(A), i_1(B))$ which shows that x_2 (x_1) must be in the first (second) component of $X \vee_p X$, a contradiction since $x \neq p$.

If $i_k(A)$ is a subset of the second component of $X \vee_p X$ and $i_k(B)$ is a subset of the first component of $X \vee_p X$, then, similarly as above, we get a contradiction since $x \neq p$.

Hence $i_k(A)$ and $i_k(B)$ cannot be in a different component of $X \vee_p X$. So if $(\{x\},\{p\}) \in \delta$, then $x = p$.

Conversely, suppose that for each $x \neq p$, $(\{x\},\{p\}) \notin \delta$. We need to show that (X,δ) is $\operatorname{Pre} T_2'$ at p, i.e. by Definitions 2.3, 2.5, 3.1 and Remark 3.1, (a) and (b) above are equivalent. We first show that (a) implies (b). Let $U\,\delta'\,V$ (δ' is a proximity structure on the wedge), i.e. $\pi_i S_p(U)\,\delta\,\pi_i S_p(V)$, $i = 1,2$.

If $i_k(A) \subseteq U$ is a subset of the first component of $X \vee_p X$ and $i_k(B) \subseteq V$ is a subset of the second component of $X \vee_p X$, then $\{x_1\} \subseteq i_k(A)$ and $\{x_2\} \subseteq i_k(B)$. By condition (P2) of Definition 2.1 it is sufficient to take "equality" instead of "subset". It follows that $\pi_1 S_p(\{x_1\})\,\delta\,\pi_1 S_p(\{x_2\}) = \pi_1(\{(x,x)\})\,\delta\,\pi_1(\{(p,x)\}) = \{x\}\,\delta\,\{p\}$, i.e. $(\{x\},\{p\}) \in \delta$. Since $(\{x\},\{p\}) \notin \delta$ (by assumption), $(\{x_1\},\{x_2\}) \notin \delta'$ by condition (P2) of Definition 2.1.

The case when $i_k(A) \subseteq U$ is a subset of the second component of $X \vee_p X$ and $i_k(B) \subseteq V$ is a subset of the first component of $X \vee_p X$ can be handled similarly. Hence, $i_k(A)$ and $i_k(B)$ cannot be in a different component of $X \vee_p X$.

If $i_k(A) \subseteq U$ and $i_k(B) \subseteq V$ are in both components of $X \vee_p X$, then $U \supseteq i_k(A) \supseteq \{x_1,x_2\}$ and $V \supseteq i_k(B) \supseteq \{x_1,x_2\}$.

If $i_k(A) \subseteq U$ is a subset of the first component of $X \vee_p X$ and $i_k(B) \subseteq V$ is a subset of both components of $X \vee_p X$, then $U \supseteq i_k(A) \supseteq \{x_1\}$ and $V \supseteq i_k(B) \supseteq \{x_1,x_2\}$.

If $i_k(A) \subseteq U$ is a subset of both components of $X \vee_p X$ and $i_k(B) \subseteq V$ is a subset of the second component of $X \vee_p X$, then $U \supseteq i_k(A) \supseteq \{x_1,x_2\}$ and $V \supseteq i_k(B) \supseteq \{x_2\}$.

If $i_k(A) \subseteq U$ and $i_k(B) \subseteq V$ are in the first component of $X \vee_p X$, then $U \supseteq i_k(A) \supseteq \{x_1\}$ and $V \supseteq i_k(B) \supseteq \{x_1\}$. Similarly, if $i_k(A) \subseteq U$ and $i_k(B) \subseteq V$ are in the second component of $X \vee_p X$, then $U \supseteq i_k(A) \supseteq \{x_2\}$ and $V \supseteq i_k(B) \supseteq \{x_2\}$.

If $(\{x_i\},\{x_i\}) \in \delta'$, then $\pi_1 S_p(\{x_1\})\,\delta\,\pi_1 S_p(\{x_1\}) = \{x\}\,\delta\,\{x\}$, i.e. $(\{x\},\{x\}) \in \delta$, $\pi_2 S_p(\{x_1\})\delta\pi_2 S_p(\{x_1\}) = \{x\}\delta\{x\}$, i.e. $(\{x\},\{x\}) \in \delta$ and $\pi_1 S_p(\{x_2\})\delta\pi_1 S_p(\{x_2\}) = \{p\}\,\delta\,\{p\}$, i.e. $(\{p\},\{p\}) \in \delta$, $\pi_2 S_p(\{x_2\})\,\delta\,\pi_2 S_p(\{x_2\}) = \{x\}\,\delta\,\{x\}$, i.e. $(\{x\},\{x\}) \in \delta$.

It follows that $(i_k(A), i_k(B)) \supseteq (\{x_i\},\{x_i\})$, $i = 1,2$, i.e. $i_k(A) \subseteq U$ and $i_k(B) \subseteq V$ are in the first or in the second component or in both components of $X \vee_p X$. So there exists a pair $x, y \in X$ such that $\{x\}\,\delta\,\{y\}$ and $i_k\{x\} = x_k \in U$ and $i_k\{y\} = y_k \in V$ for some $k = 1$ or 2. This shows that (a) implies (b).

We now show that (b) implies (a). Suppose (b) holds. We need to show that for any sets U, V on the wedge $p_1 U\,\delta\,p_1 V$ and $\bigtriangledown_p U\,\delta\,\bigtriangledown_p V$, i.e. $\pi_i S_p(U)\,\delta\,\pi_i S_p(V)$, $i = 1,2$, there exists a pair $x, y \in X$ such that $\{x\}\,\delta\,\{y\}$ and $i_k\{x\} = x_k \in U$ and $i_k\{y\} = y_k \in V$ for some $k = 1$ or 2. By using similar argument as above, we must have $(i_k\{x\}, i_k\{y\}) \supseteq (\{x_i\},\{x_i\})$, $i = 1,2$. For $i = 1$ if $(\{x_1\},\{x_1\}) \in \delta'$, then $\pi_1 S_p(\{x_1\})\,\delta\,\pi_1 S_p(\{x_1\}) = \{x\}\,\delta\,\{x\}$, i.e. $(\{x\},\{x\}) \in \delta$, $\pi_2 S_p(\{x_1\})\,\delta\,\pi_2 S_p(\{x_1\}) = \{x\}\,\delta\,\{x\}$, i.e. $(\{x\},\{x\}) \in \delta$. For $i = 2$ if $(\{x_2\},\{x_2\}) \in \delta'$, then $\pi_1 S_p(\{x_2\})\,\delta$

$\pi_1 S_p(\{x_2\}) = \{p\}\ \delta\ \{p\}$, i.e. $(\{p\},\{p\}) \in \delta$, $\pi_2 S_p(\{x_2\})\ \delta\ \pi_2 S_p(\{x_2\}) = \{x\}\ \delta\ \{x\}$, i.e. $(\{x\},\{x\}) \in \delta$. Hence $\pi_i S_p(U) \delta \pi_i S_p(V)$, $i = 1, 2$. This shows that (b) implies (a).

Hence (X, δ) is $\operatorname{Pre} T_2'$ at p. □

Theorem 3.4. *Let (X, δ) be an Efremovich proximity space and $p \in X$. (X, δ) is $\overline{T}_2$ at p if and only if for each $x \neq p$, $(\{x\},\{p\}) \notin \delta$.*

P r o o f. It follows from Definition 3.1 and Theorems 3.1, 3.2. □

Theorem 3.5. *Let (X, δ) be an Efremovich proximity space and $p \in X$. (X, δ) is T_2' at p if and only if for each $x \neq p$, $(\{x\},\{p\}) \notin \delta$.*

P r o o f. It follows from Definition 3.1 and Theorems 3.1, 3.3. □

4. Regular objects at a point

In this section, the characterizations of each of the various notions of the separation property T_3 at a point p are given in the topological category of proximity spaces Prox.

Let B be set and $p \in B$. The infinite wedge product $\bigvee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times \ldots$ be the countable cartesian product of B. Define $A_p^\infty \colon \bigvee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, \ldots, p, x, p, \ldots)$, where x_i is in the ith component of the infinite wedge and x is in the ith place in $(p, p, \ldots, p, x, p, \ldots)$ (infinite principal p-axis map), and $\bigtriangledown_p^\infty \colon \bigvee_p^\infty B \to B$ by $\bigtriangledown_p^\infty(x_i) = x$ for all $i \in I$ (infinite fold map), see [2], [3].

Note also that A_p^∞ is the unique map arising from the multiple pushout of $p \colon 1 \to B$ for which $A_p^\infty i_j = (p, p, \ldots, p, \mathrm{id}, p, \ldots) \colon B \to B^\infty$, where the identity map id is in the jth place (see [9]).

Definition 4.1 (cf. [2], [3])**.** Let $\mathcal{U} \colon \mathcal{E} \to \mathrm{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$-sink $q \colon \mathcal{U}(X) = B \to B/F = (B \setminus F) \cup \{*\}$, where q is the epi map that is the identity on $B \setminus F$ identifying F with a point $\{*\}$.

Let p be a point in B.

(1) p is closed if and only if the initial lift of the $\mathcal{U}$-source

$$\left\{ A_p^\infty \colon \bigvee_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty \text{ and } \bigtriangledown_p^\infty \colon \bigvee_p^\infty B \to \mathcal{U}\mathcal{D}(B) = B \right\}$$

is discrete.

(2) $F \subset X$ is closed if and only if $\{*\}$, the image of F, is closed in X/F or $F = \emptyset$.
(3) $F \subset X$ is strongly closed if and only if X/F is T_1 at $\{*\}$ or $F = \emptyset$.
(4) If $B = F = \emptyset$, then we define F to be both closed and strongly closed.
(5) X is $\overline{T}_3$ at p if and only if X is T_1 at p and X/F is $\mathrm{Pre}\,\overline{T}_2$ at p for all closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.
(6) X is T_3' at p if and only if X is T_1 at p and X/F is $\mathrm{Pre}\,T_2'$ at p for all closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.
(7) X is $S\overline{T}_3$ at p if and only if X is T_1 at p and X/F is $\mathrm{Pre}\,\overline{T}_2$ at p for all strongly closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.
(8) X is ST_3' at p if and only if X is T_1 at p and X/F is $\mathrm{Pre}\,T_2'$ at p for all strongly closed $F \neq \emptyset$ in $\mathcal{U}(X)$ missing p.

R e m a r k 4.1. Note that for the category Top of topological spaces we have:

(1) The notion of closedness coincides with the usual closedness, see [2], and $F \subset X$ is strongly closed if and only if F is closed and for each $x \in X$ with $x \notin F$ there exists a neighborhood of F missing x, see [2]. If a topological space X is T_1, then the notions of closedness and strong closedness coincide, see [2]. The notion of (strong) closedness forms closure operators in the sense of Dikranjan and Giuli (see [14]) in some well-known topological categories (see [8], [10], [13]).
(2) A topological space X is $\overline{T}_3$ (T_3') at $p \in X$ if and only if for each $x \in X$ with $x \neq p$ there exists a neighborhood of x missing p and a neighborhood of p missing x, and for any nonempty closed set F missing p there exist disjoint open sets containing F and p, see [2].
(3) A topological space X is $S\overline{T}_3$ (ST_3') at p if and only if for each $x \in X$ with $x \neq p$ there exists a neighborhood of x missing p and a neighborhood of p missing x, and for any nonempty closed set F missing p for which each point x not in F there exists a neighborhood of F missing x (i.e. F is a strongly closed set), there exist disjoint open sets containing F and p, see [2].

The following result is given in [21].

Theorem 4.1. *Let (X, δ) be an Efremovich proximity space and $p \in X$.*

(1) *$\{p\}$ is closed in X if and only if for any $B \subset X$, if $\{p\}\ \delta\ B$, then $p \in B$.*
(2) *$\emptyset \neq F \subset X$ is closed or strongly closed if and only if $x \in F$ whenever $\{x\}\ \delta\ F$ for all $x \in X$.*

Theorem 4.2. *Let (X, δ) be an Efremovich proximity space and $p \in X$. (X, δ) is* $\mathrm{Pre}\,T_2'$ *at p, then $(X/F, \delta^*)$ is* $\mathrm{Pre}\,T_2'$ *at p.*

P r o o f. Suppose (X,δ) is $\mathrm{Pre}\,T_2'$ at p. Let a and p be any distinct pair of points in X/F. By Theorem 3.3, we only need to show that $(\{a\},\{p\}) \notin \delta^*$, where δ^* is the structure on X/F induced by q.

Let $p = *$. Suppose that $a \neq *$. By definition of q map, there exist $a \in X$ and $F \subset X$ such that $q(a) = a$ and $q(c) = *$ for any $c \in F$. Since $a \neq c$ for any $c \in F$ ($a \notin F$) and (X,δ) is $\mathrm{Pre}\,T_2'$ at p, then $\{a\}\,\overline{\delta}\,\{c\}$. By condition (P2) of Definition 2.1 we obtain $\{a\}\,\overline{\delta}\,F$. Then we have $\{a\}\,\overline{\delta}\,F = q^{-1}(\{a\})\,\overline{\delta}\,q^{-1}(\{*\})$. It follows that by p-neighborhood relation definition and Definition 2.4, for each binary rational s in $[0,1]$ there is some $C_s \subset X/F$ such that $C_0 = \{a\}$, $C_1 = \{*\}^c$ and $s < t$ implies $q^{-1}(C_s) \ll_\delta q^{-1}(C_t) = q^{-1}(\{a\}) \ll_\delta (q^{-1}(\{*\}))^c = q^{-1}(\{a\}) \ll_\delta q^{-1}(\{*\}^c)$ if and only if $\{a\} \ll^* \{*\}^c$. Hence $\{a\}\,\overline{\delta}^*\,\{*\}$, i.e. $(\{a\},\{*\}) \notin \delta^*$.

Let $a \neq p \neq *$. By definition of q map, there exists a pair $a, p \in X$ such that $q(a) = a$ and $q(p) = p$. In this case q map can be considered as one-to-one map. Suppose that $\{a\}\delta^*\{p\}$. By definition of q map and Definition 2.4, we have $\{a\}\,\delta^*\,\{p\}$ if and only if $q^{-1}(\{a\})\,\delta\,q^{-1}(\{p\}) = \{a\}\,\delta\,\{p\}$. But $\{a\}\,\overline{\delta}\,\{p\}$ since (X,δ) is $\mathrm{Pre}\,T_2'$ at p. Hence $\{a\}\,\overline{\delta}^*\,\{p\}$ i.e. $(\{a\},\{p\}) \notin \delta^*$.

Consequently, for each of the distinct points a and p in X/F we have $(\{a\},\{p\}) \notin \delta^*$. Hence by Theorem 3.3, $(X/F,\delta^*)$ is $\mathrm{Pre}\,T_2'$ at p. □

Corollary 4.1. *Let (X,δ) be an Efremovich proximity space and $p \in X$. Then the following statements are equivalent:*

(1) *(X,δ) is $\overline{T}_3$ at p.*
(2) *(X,δ) is T_3' at p.*
(3) *(X,δ) is $S\overline{T}_3$ at p.*
(4) *(X,δ) is ST_3' at p.*
(5) *For each $x \in X$ with $x \neq p$, $(\{x\},\{p\}) \notin \delta$.*

P r o o f. It follows from Theorems 3.1 (1) and 4.2. □

5. Generalized separation properties and relationships

Let B be a nonempty set, $B^2 = B \times B$ be cartesian product of B with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of B^2 identified along the diagonal, i.e. the result of pushing out Δ along itself. A point (x,y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x,y)_1$ (or $(x,y)_2$) if (x,y) is in the first (or second) component of $B^2 \vee_\Delta B^2$. Clearly $(x,y)_1 = (x,y)_2$ if and only if $x = y$, see [2].

The principal axis map $A\colon\ B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x,y)_1 = (x,y,x)$ and $A(x,y)_2 = (x,x,y)$. The skewed axis map $S\colon\ B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x,y)_1 =$

(x, y, y) and $S(x, y)_2 = (x, x, y)$ and the fold map $\bigtriangledown\colon B^2 \vee_\Delta B^2 \to B^2$ is given by $\bigtriangledown(x, y)_i = (x, y)$ for $i = 1, 2$. Note that $\pi_1 S = \pi_{11} = \pi_1 A$, $\pi_2 S = \pi_{21} = \pi_2 A$, $\pi_3 A = \pi_{12}$ and $\pi_3 S = \pi_{22}$, where $\pi_k\colon B^3 \to B$ is the kth projection, $k = 1, 2, 3$, and $\pi_{ij} = \pi_i + \pi_j\colon B^2 \vee_\Delta B^2 \to B$ for $i, j \in \{1, 2\}$, see [2].

Definition 5.1 (cf. [2], [7], [9])**.** Let $\mathcal{U}\colon \mathcal{E} \to$ Set be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.

(1) X is $\overline{T}_0$ if and only if the initial lift of the $\mathcal{U}$-source $\{A\colon B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\bigtriangledown\colon B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.

(2) X is T_0' if and only if the initial lift of the $\mathcal{U}$-source $\{\mathrm{id}\colon B^2 \vee_\Delta B^2 \to \mathcal{U}(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\bigtriangledown\colon B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2\colon \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$, i_1 and i_2 are the canonical injections and $\mathcal{D}(B^2)$ is the discrete structure on B^2.

(3) X is T_1 if and only if the initial lift of the $\mathcal{U}$-source $\{S\colon B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\bigtriangledown\colon B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete.

(4) X is $\mathrm{Pre}\,\overline{T}_2$ if and only if the initial lift of the $\mathcal{U}$-sources $A\colon B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $S\colon B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ agree.

(5) X is $\mathrm{Pre}\,T_2'$ if and only if the initial structure on $B^2 \vee_\Delta B^2$ induced from $\mathcal{U}$-source $S\colon B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and the final structure on $B^2 \vee_\Delta B^2$ induced from $\mathcal{U}$-sink $\{i_1, i_2\colon \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ coincide.

(6) X is $\overline{T}_2$ if and only if X is $\overline{T}_0$ and $\mathrm{Pre}\,\overline{T}_2$.

(7) X is T_2' if and only if X is T_0' and $\mathrm{Pre}\,T_2'$.

The following result is given in [22].

Theorem 5.1.

(1) *All Efremovich proximity spaces are* T_0' *and* $\mathrm{Pre}\,\overline{T}_2$.

(2) *An Efremovich proximity space* (X, δ) *is* $\overline{T}_0$ *(or* T_1, $\mathrm{Pre}\,T_2'$, T_2', $\overline{T}_2$*) if and only if for each distinct pair* x *and* y *in* X, $(\{x\}, \{y\}) \notin \delta$.

Definition 5.2 ([23])**.** An Efremovich proximity space (X, δ) is said to be a T_2-*space* (*Hausdorff*) if $x\,\delta\,y$ for $x, y \in X$ implies that $x = y$.

We can infer the following results.

R e m a r k 5.1.

(1) For an arbitrary topological category,

(i) by Theorem 2.7 (1) of [11], $\overline{T}_0$ implies T_0' but the converse implication is generally not true,

(ii) by Theorem 2.7 of [7], if X is $\mathrm{Pre}\,T_2'$ (or T_2', T_3', ST_3'), then X is $\mathrm{Pre}\,\overline{T}_2$ (or $\overline{T}_2$, $\overline{T}_3$, $S\overline{T}_3$) but the converse implication is generally not true.

(2) By Theorem 2.8 of [7], if $\mathcal{U}\colon \mathcal{E} \to \mathrm{Set}$ is normalized, then $\overline{T}_i$ (or $S\overline{T}_i$, i=2,3) implies $\overline{T}_i$ (or $S\overline{T}_i$, i=2,3) at p.

(3) By Theorem 1.5 and Remark 1.8 of [5], for the category Top of topological spaces $\overline{T}_0$ at p and T_0' at p (or $\mathrm{Pre}\,\overline{T}_2$ at p and $\mathrm{Pre}\,T_2'$ at p) are equivalent. In general, by Parts (1) and (2), $\overline{T}_0$ at p (or $\mathrm{Pre}\,T_2'$ at p) implies T_0' at p (or $\mathrm{Pre}\,\overline{T}_2$ at p). But the converse implication is in general not true. For example, let $X = \{a, b\}$ and $\delta = \{(X,X), (\{a\},\{a\}), (\{b\},\{b\}), (X,\{a\}), (\{a\},X), (X,\{b\}), (\{b\},X), (\{a\},\{b\}), (\{b\},\{a\})\}$. Then, by Theorems 3.1–3.3, an Efremovich proximity space (X,δ) is T_0' at a and $\mathrm{Pre}\,\overline{T}_2$ at a but it is neither $\overline{T}_0$ at a nor $\mathrm{Pre}\,T_2'$ at a since $(\{a\},\{b\}) \in \delta$ with $a \neq b$.

(4) By Theorems 3.1, 3.3–3.5 and Corollary 4.1, the following statements are equivalent:

(a) (X,δ) is $\overline{T}_0$ at $p \in X$.
(b) (X,δ) is T_1 at p.
(c) (X,δ) is $\mathrm{Pre}\,T_2'$ at p.
(d) (X,δ) is $\overline{T}_2$ at p.
(e) (X,δ) is T_2' at p.
(f) (X,δ) is $\overline{T}_3$ at p.
(g) (X,δ) is T_3' at p.
(h) (X,δ) is ST_3' at p.
(i) For each point $x \in X$ with $x \neq p$, $(\{x\},\{p\}) \notin \delta$.

(5) By Theorem 5.1 and Definition 5.2, the following statements are equivalent:

(a) (X,δ) is $\overline{T}_0$.
(b) (X,δ) is T_1.
(c) (X,δ) is $\mathrm{Pre}\,T_2'$.
(d) (X,δ) is $\overline{T}_2$.
(e) (X,δ) is T_2'.
(f) (X,δ) is $\overline{T}_3$.
(g) (X,δ) is T_3'.
(h) (X,δ) is ST_3'.
(i) (X,δ) is T_2.
(j) For each distinct pair of points x and y in X, $(\{x\},\{y\}) \notin \delta$.

(6) By Theorems 3.3–3.5 and Theorem 5.1, (X,δ) is $\mathrm{Pre}\,T_2'$ (or $\overline{T}_2$, T_2') at p for all $p \in X$ if and only if (X,δ) is $\mathrm{Pre}\,T_2'$ (or $\overline{T}_2$, T_2').

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